Lecture 29:

Last time: We talked more about normal variables.

$$\mathcal{N}(\mu, \sigma^2)$$

Mean — μ; $(\text{s.t.d.})^2 = \text{variance}$ — σ^2.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

- **Key facts**: $\mathcal{N}(\mu, \sigma) = \mu + \sigma\,\mathcal{N}(0,1)$
- $$\begin{aligned} F_{\mathcal{N}(\mu,\sigma)}(a) &= P(\mathcal{N}(\mu,\sigma) \le a) \\ &= P(\mu + \sigma\,\mathcal{N}(0,1) \le a) \\ &= P\left(\mathcal{N}(0,1) \le \frac{a-\mu}{\sigma}\right) =: \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$

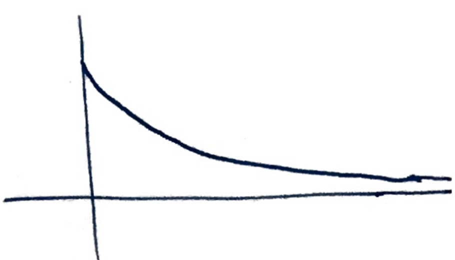
Φ is the cumulative distribution function of standard normal $\mathcal{N}(0,1)$.

Exponential random variables:

usually used for amount of time until something happens.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

"exponentially distributed with ~~for~~ parameter λ".



Cumulative distribution function

$$F(a) = P(X \le a) = \int_{0}^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^a = 1 - e^{-\lambda a}$$

mean:

$$\mathbb{E}[X] = \int_0^\infty x\lambda e^{-\lambda x}\,dx = -xe^{-\lambda x}\Big|_0^\infty - \int_0^\infty (-e^{-\lambda x})\,dx =$$

$u = x \qquad du = 1$

$dv = \lambda e^{-\lambda x}dx \qquad v = -e^{-\lambda x}$

$$= 0 - 0 + \frac{1}{\lambda} = \boxed{\frac{1}{\lambda}}$$

$\mathrm{Var}[X] = \frac{1}{\lambda^2}$ (similar calculation)

Ex: Suppose the length of a phone call is modeled by an exponential random variable. On average, the call takes 10 minutes. What is the probability that a call takes more than 10 minutes?

$f(x) = \lambda e^{-\lambda x}$ we know $\mathbb{E}[X] = 10 = 1/\lambda$ so $\lambda = \frac{1}{10}$.

$f(x) = \frac{1}{10}e^{-1/10\,x}$

$$P(X>10) = 1 - \int_{-\infty}^{10} \frac{1}{10}e^{-1/10\,x}\,dx = 1 - \left(1 - e^{-1/10 \cdot 10}\right)$$

$1 - F(10)$

$$= 1/e \approx 1/2.7$$

Ex2: Same situation: If I already ~~wait~~ took 10 minutes for my call, what's the probability that my call takes more than 20 minutes.

(Let's do this in general. If I ~~waited~~ took a minutes, what is the probability that I ~~wait~~ take b minutes ($b \geq a$).)

$$P(X \geq a) = 1 - F(a) = 1 - \left(1 - e^{-1/10\, a}\right) = e^{-1/10\, a}$$

$$P(X \geq b) = e^{-1/10\, b}$$

$$P(X \geq b \mid X \geq a) = \frac{P\big(\overset{X \geq b}{\cancel{X \geq b \text{ and } X \geq a}}\big)}{P(X \geq a)} = \frac{e^{-1/10\, b}}{e^{-1/10\, a}} = e^{-1/10 (b-a)}$$

Remark: This is what's clever about the exponential random variable. At each point, the amount of more time it's going to take is independent of how long it already took!

(makes more sense for things other than phone calls)

Some other random variables:

(usually used for amount of time it takes until n events occur $(\alpha = n)$

Gamma distribution: (with parameters (α, λ)) $\lambda > 0$.

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

when $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha - 1}\, dy$ "Gamma function".

Gamma function is ~~so~~ a continuous version of factorial.

$$\Gamma(n) = (n-1)!$$

$$E[X] = \frac{\alpha}{\lambda}$$

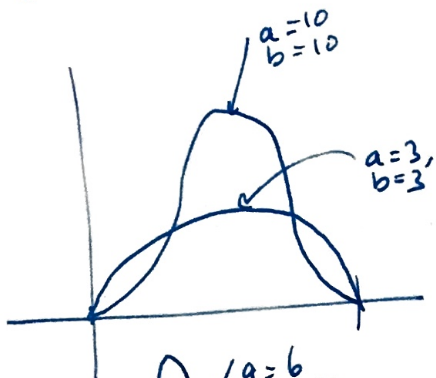
(if you use integration by parts on $\Gamma(\alpha)$ you get this)

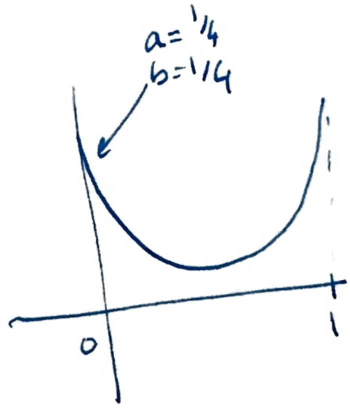
Beta-distribution:

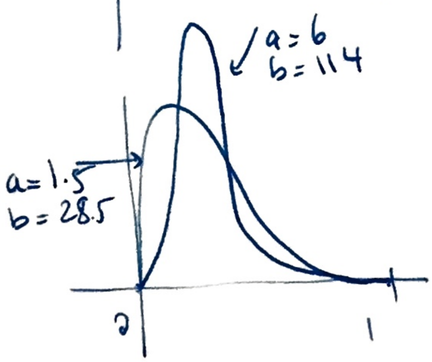
$$f(x) = \begin{cases} \dfrac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{where } B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} \, dx. = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

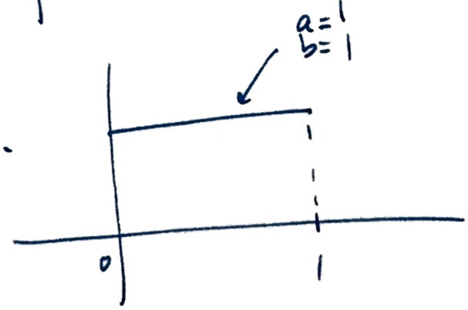
→ the nice thing about it is that it makes it easy to make a random variable with values in finite interval.







very flexible.



$$\mathbb{E}[X] = \frac{a}{a+b}$$

There are other nice things about Beta distribution too,
look up "Beta distribution"
"maximum likelihood estimation"
"bernouilli".