Lecture 27: - HW new due date
- Course evaluations

Last time: Continued continuous r.v.

- Expectation: $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx$.

  also $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$.

- cumulative distribution function: $F(a) = \int_{-\infty}^{a} f(x) dx$.

$F(a) = P(X \leq a)$.

Ex: I am going to class and my travel time after I depart is a random variable X with density function $f(x)$.

If I am s minutes early, it costs me cs

If I am s minutes late, it costs me ks

When should I leave to minimize my cost

Solution: X = travel time

$C_t(X)$ = cost if I leave t minutes early.

$$C_t(X) = \begin{cases} c(t-X) & \text{if } X \leq t \quad \text{I'm early} \\ k(X-t) & \text{if } X > t \quad \text{I'm late.} \end{cases}$$

We want to compute $\mathbb{E}[C_t(X)]$ and pick the t with the least expected cost

$$\mathbb{E}[C_t(X)] = \int_0^{\infty} C_t(X) f(x) dx.$$

$$= \int_0^\infty C_t(X) f(x)\,dx = \int_0^t c(t-x) f(x)\,dx + \int_t^\infty k(x-t) f(x)\,dx$$

$$= ct \int_0^t f(x)\,dx - c\int_0^t x f(x)\,dx + k\int_t^\infty x f(x)\,dx - tk\int_t^\infty f(x)\,dx.$$

we can't simplify more, ~~but~~ (without knowing $f(x)$) but what we are looking for is minimum of $\mathbb{E}[C_t(X)]$ so we should take the derivative.

$$\frac{d}{dt}\mathbb{E}[C_t(X)] = ctf(t) + c\underbrace{\int_0^t f(x)\,dx}_{F(t)} - ctf(t) - ktf(t) + ktf(t) - k(1-F(t)) = 0$$

$$= (k+c)F(t) - k = 0.$$

so the best t is when $F(t) = \dfrac{k}{k+c}$.

<u>Remark</u>: All the things we know about expectation are still true:

$$\mathbb{E}[aX+b] = a\mathbb{E}[X] + b$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

<u>Recall</u>: Uniform random variable $U_{[\alpha,\beta]}$ $\quad f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } x\in[\alpha,\beta] \\ 0 & \text{otherwise.} \end{cases}$

we previously computed $\mathbb{E}[U_{[\alpha,\beta]}] = \dfrac{\alpha+\beta}{2}$

$$Var[U_{[\alpha,\beta]}] = \frac{(\beta-\alpha)^2}{12}$$

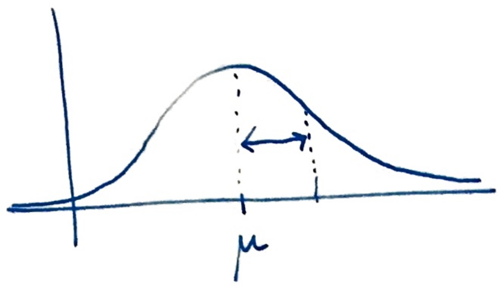
Normal random variables: $X = \mathcal{N}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$\mu = \mathbb{E}[X] = \text{mean}$

$\sigma = \sqrt{Var(X)}$

= "standard deviation".



~ 68% of the probability is within $(\mu-\sigma, \mu+\sigma)$

~ 95% is within $(\mu-2\sigma, \mu+2\sigma)$

Is observed a lot in real life.
(grades, height, weight, error amounts, ...)

Because of this theorem:

Central limit theorem: Let $\overset{=X}{X_i}$ be a probability distribution with mean $= \mu$ and standard deviation σ, $(X = X_1 = X_2 = \dots$ all equal$)$

and let $Y_n = \dfrac{X_1 + X_2 + \dots + X_n}{n}$ be the average of the X_i,

then, as $n \to \infty$, Y_n approaches the distribution

$$\mathcal{N}\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

This is proven in 130B or 130C.

The point: Things that are the average (or combination) of many other things tend to have normal distribution (eg in a test, your score is a combination of your scores on every problem...)

total probability:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$y=(x-\mu)/\sigma \quad dy=\frac{dx}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{-\infty}^{\infty} e^{-y^2/2}\,\cancel{\sigma}\,dy \overset{?}{=} 1$$

So we must show:

$$I = \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$
$dx\,dy = r\,dr\,d\theta$

$$I^2 = \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\,r\,d\theta\,dr$$

$$= 2\pi \int_0^{\infty} e^{-r^2/2}\,r\,dr$$

$$= -2\pi e^{-r^2/2}\Big|_0^{\infty}$$

$$= 2\pi$$

Expectation: $\mathbb{E}[\mathcal{N}(\overset{0}{\cancel{\mu}},\overset{1}{\cancel{\sigma^2}})] = \int_{-\infty}^{\infty} x \cdot e^{-x^2/2}\,dx$

$$= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\Big|_{-\infty}^{\infty} = 0$$

Cool fact: If $\mathcal{N}(\mu,\sigma^2) = X$ $\longrightarrow$ $\mathbb{E}[\mathcal{N}(\mu,\sigma^2)]$

then $aX+b = \mathcal{N}(\mu+b, a^2\sigma^2)$ $\quad = \mathbb{E}[\sigma\mathcal{N}(0,1)+\mu]$

$= \sigma\,\mathbb{E}[\mathcal{N}(0,1)]+\mu = \sigma\cdot 0+\mu = \mu.$

As we said at the beginning:

$$Var(\mathcal{N}(\mu, \sigma^2)) = \sigma^2$$

(similar, use integration by parts, book page 189.)