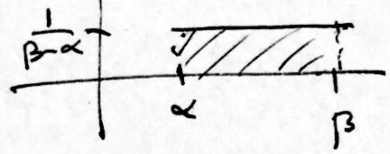
Lecture 26:

Last time: we talked about continuous random variables.

- $P(X \in [a,b]) = \int_a^b f(x)\,dx$ ← density function

  so $\int_{-\infty}^{\infty} f(x)\,dx = P(X \in (-\infty,\infty)) = 1$

ex: uniform distribution $X = U_{[\alpha,\beta]}$



density function
$$f(x) = \frac{1}{\beta-\alpha}$$ ← constant function.

- cumulative distribution of X is:
$$F(a) = P(X \in (-\infty, a]) = \int_{-\infty}^{a} f(x)\,dx.$$

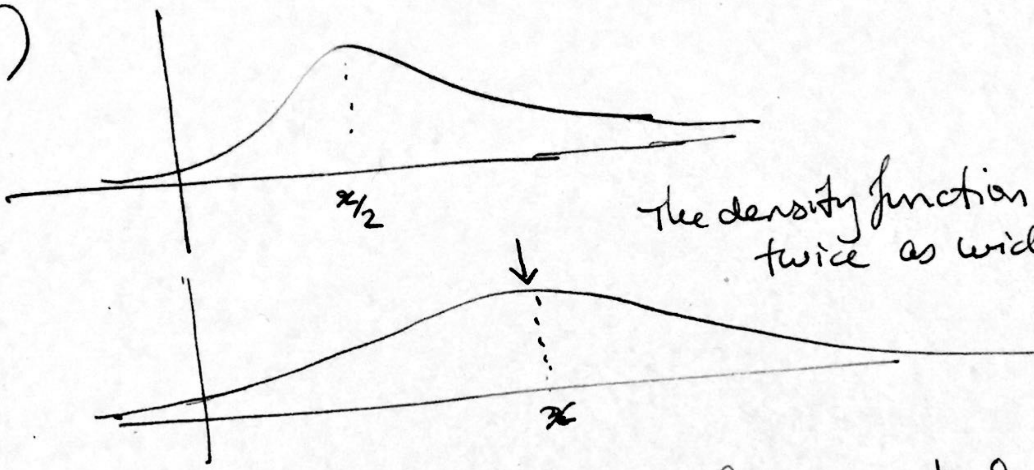
so: $\frac{d}{da}F(a) = f(a)$

Q: What is the density function of $Y = 2X$

" " " ~~dis~~ cumulative distribution function of $Y = 2X$

in terms of the density and ~~dist~~ cumulative distribution functions f_X and F_X of X?

1)



The density function of f_Y should be twice as wide, but we should also rescale it so that the integral stays 1.

ie. $$f_Y(x) = \frac{1}{2} f_X\left(\frac{x}{2}\right)$$

but there is an easier way after doing 2).

2) $$F_Y(a) = P(Y \leq a)$$
$$= P(2X \leq a)$$
$$= P(X \leq a/2)$$
$$= F_X(a/2)$$

back to 1): take derivative of both.

$$f_Y(a) = \frac{d}{da} F_X\left(\frac{a}{2}\right) = \frac{1}{2} f_X\left(\frac{a}{2}\right)$$ by chain rule.

Expectation: for discrete, we had

$$\mathbb{E}[X] = \sum_i x_i \, p(x_i) = \sum_x x \, P(X=x)$$

for continuous:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$$

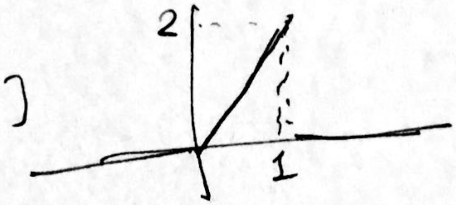
What about variance?

discrete: $\operatorname{Var} X = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.   mean $= \mathbb{E}[X]$.

so we can use the same:

<u>Ex:</u> Let X have density function:

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$



$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\cancel{-\infty}\ 0}^{1\,\cancel{\infty}} x \cdot 2x = \int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}.$$

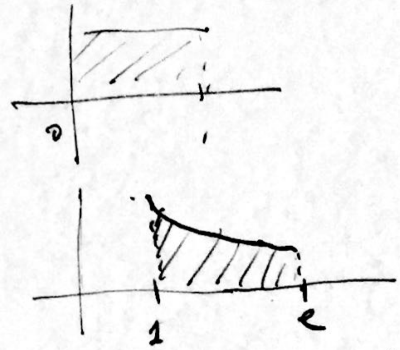
<u>Ex:</u> $X = U_{[0,1]}$ uniform distribution.

what is $\mathbb{E}[e^X] = ?$

<u>solution:</u> we need to find the density function of e^X.

we can do that by finding the cumulative distrib. function first. $Y = e^X$

$$F_Y(x) = P(Y \le x)$$
$$= P(e^X \le x)$$
$$= P(X \le \log(x))$$
$$= \int_0^{\log x} f(y)\,dy = \log(x)$$



$\xrightarrow{}$ assuming $\log(x) \in [0,1]$ ie $x \in [1,e]$

1

for $x \in [1,e]$ and 0 otherwise

so $f_Y(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\log x = \frac{1}{x}$

so $\mathbb{E}[Y] = \mathbb{E}[e^X] = \int_{-\infty}^{\infty} x \cdot f_Y(x)\,dx = \int_1^e \underset{x \cdot \frac{1}{x}}{1}\,dx = e - 1$

We could have done this much more easily by using:

Proposition: X continuous r.v. with density $f(x)$.
g any real function:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

So for our previous example:

$$\mathbb{E}[e^X] = \int_0^1 e^x\, dx = e - 1.$$

Ex: I am going to class and my travel time after I depart is a random variable X with density function $f(x)$.
If I am s minutes early, it costs me cs
" " " s " late, " " " ks.

When should I leave to minimize expected cost?

Solution: X = travel time.
If I leave t minutes early; the cost is $C_t(X)$

$$C_t(X) = \begin{cases} c(t-X) & \text{if } X \le t \quad \checkmark \text{ I'm early} \\ k(X-t) & \text{if } X > t \quad \checkmark \text{ I'm late!} \end{cases}$$

Let's compute:

$$\mathbb{E}[C_t(X)] = \int_0^{\infty} C_t(x) f(x)\, dx.$$

$$\int_0^\infty C_t(x) f(x)\,dx = \int_0^t c(t-x) f(x)\,dx + \int_t^\infty k(x-t) f(x)\,dx$$

$$= ct\int_0^t f(x)\,dx - c\int_0^t x f(x)\,dx + k\int_t^\infty x f(x)\,dx - tk\int_t^\infty f(x)\,dx.$$

we can't go faster without knowing f, but since we are trying to minimize $\mathbb{E}[C_t(X)]$, we can look at derivative:

$$\frac{d}{dt}\mathbb{E}[C_t(X)] = ctf(t) + c\underbrace{\int_0^t f(x)\,dx}_{F(t)} - ctf(t)$$

$$- ktf(t) + ktf(t) - k(1-F(t))$$

$$= (k+c)F(t) - k. = 0.$$

So ~~the~~ best time to leave is exactly when $F(t) = \dfrac{k}{k+c}$

eg. If $k=c$, we would leave at the mean of X (minutes early) because that's when $F(t) = \frac{1}{2}$