Lecture 24:

Other random variables less popular than binomial or Poisson.

Geometric r.v.

Suppose independent trials are performed, each successful with probability p, until a success occurs.

$$X = \text{" \# of trials performed "}$$

$$P(X=n) = (1-p)^{n-1}\, p \qquad n = 1, 2, \ldots$$

Let's look at the total probability:

$$\sum_{n=1}^{\infty} P(X=n) = \sum_{n=1}^{\infty} (1-p)^{n-1} p = p \sum_{j=0}^{\infty} (1-p)^{j} = p \cdot \frac{1}{1-(1-p)} = 1.$$

$j = n-1$

geometric series

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

Ex: I decided that I'm going to play the lotto, (winning probability p) every week once, until I win and then retire. What is the expected number of times I would play? (expected number of weeks until retirement)

Before we solve, let's find the expected number in general:

$$\mathbb{E}[X] = ?$$

Expectation of geometric r.v.

Let $q = 1-p$.

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \underbrace{i q^{i-1}}_{\text{looks like derivative}} p$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$= p \sum_{i=1}^{\infty} i q^{i-1}$$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{-1 \cdot (-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$= \frac{p}{(\underbrace{1-q}_{p})^2} = \frac{p}{p^2} = \boxed{\frac{1}{p}}$$

number of (expected) weeks till retirement: $\frac{1}{p} \approx 13.000.000$ weeks in U.K. lottery (we computed before)

We can do the series for the variance too and get:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \dots = \frac{1-p}{p^2}$$

Negative Binomial random variable:

Independent trials are performed until we get r successes.

X = "# of trials until we get r successes".

$$P(X=n) = \binom{n-1}{r-1} p^r (1-p)^{n-r} \qquad n = r, r+1, \dots$$

how many ways can we arrange the successful ones that are not the last one.

Ex: What is the probability of having r successes before m failures? each success is with probability p.

Solution: r successes must occur before $(r+m)$th trial because otherwise I am guaranteed to have m failures. So the ~~total probability~~ probability is:

$$\sum_{n=r}^{r+m-1} P(X=n) = \sum_{n=r}^{r+m+1} \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$

Expected value: we can do the sum as usual, but we can also be very clever:

observe that, if $Y = \#$ of trials until one success. (geometric)

then $X = \underbrace{Y + Y + \cdots + Y}_{r \text{ times}}$.

so $\mathbb{E}[X] = \underbrace{\mathbb{E}[Y]}_{1/p} + \cdots + \underbrace{\mathbb{E}[Y]}_{1/p} = \dfrac{r}{p}$.

Hypergeometric r.v.

I have a box with n balls, m white, $n-m$ black,

X = The number of white balls I would get if I randomly select r balls (without replacement)

$$P(X=i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

$$\mathbb{E}[X] = \frac{nm}{N} \qquad \text{(Book page 154)}$$