Lecture 23:

Last time: Poisson random variable. X_λ

$$P(X_\lambda = i) = e^{-\lambda}\frac{\lambda^i}{i!}$$

We invented Poisson r.v. by looking at binomial r.v with $n \to \infty$ and p small compared to n.

Ex: On average, out of all the people walking on 5th avenue, 3.2 people walk into Saks 5th avenue every hour. What is the probability that at most two customers walk into the store in a given hour?

solution: X = # of customers that walk in in an hour

we know $\mathbb{E}[X] = 3.2$

We want to assume $X = X_\lambda$ is Poisson.

But what is $\lambda = ?$ Pause.

Expectation of Poisson r.v.:

$$\mathbb{E}[X_\lambda] = \sum_{i=1}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!}\, i = \sum_{i=1}^{\infty} e^{-\lambda}\frac{\lambda^i}{(i-1)!} = \lambda \underbrace{\sum_{i=1}^{\infty} e^{-\lambda}\frac{\lambda^{i-1}}{(i-1)!}}_{1 = \text{total probability of } X_\lambda}.$$

removed $i=0$ since it contributes 0.

$$\mathbb{E}[X_\lambda] = \lambda.$$

$\boxed{\triangleright}$ rest of solution to problem:

$\lambda = 3.2$

$$P(X=i) = e^{-3.2} \frac{(3.2)^i}{i!}$$

$$\left.\begin{aligned} P(X=0) &= e^{-3.2} \\ P(X=1) &= e^{-3.2} \frac{3.2}{1} \\ P(X=2) &= e^{-3.2} \frac{(3.2)^2}{2!} \end{aligned}\right\} \text{sum of these.}$$

Variance of Poisson:

$$\mathbb{E}[(X-\lambda)^2] = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} (i-\lambda)^2 = \dots$$

too complicated compared to doing it X^2 first ...

$$\mathbb{E}[X^2] = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} i^2$$

$$= \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^{i-1}}{(i-1)!} \cdot \lambda \cdot i \qquad \swarrow j = i-1$$

$$= \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} \lambda(j+1)$$

$$= \lambda \Big(\underbrace{\sum e^{-\lambda} \frac{\lambda^j}{j!} j}_{\text{same computation as expectation}} + \underbrace{\sum e^{-\lambda} \frac{\lambda^j}{j!}}_{\text{1 total probability like before.}} \Big) = \lambda(\lambda+1)$$

$$\mathbb{E}[X]^2 = \lambda^2$$

So $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Ex: n men randomly select hats from a pile to which they have all thrown their hats. what is the probability that none of them select their own hat?

solution: We did this before, quite carefully using inclusion - exclusion principle and the taylor series for e^x.

The probability that each man gets his own hat is ~~approximately~~ $\frac{1}{n}$.

The issue is that each person getting their own hat are not independent random variables. For example, if 29 out of 30 people got their own hat, then 30th person must get his own hat too (not independent!).

But: let E_i = "ith person got own hat"

$$P(E_i) = 1/n$$

$$P(E_i \mid E_j) = \frac{1}{n-1}$$

So it's very close to being independent!

So we can use Poisson:

$$P(X_\lambda = i) = e^{-\lambda}\frac{\lambda^i}{i!}$$

$$\mathbb{E}[X_\lambda] \approx 1 = \lambda$$

on average 1 person should get own hat.

since n ppl. each person has $1/n$ chance.

conclusion $\lambda = 1$

$$P(X_1 = 0) = e^{-1}$$

Length of longest run:

Coin is flipped n times. Each time $P(H) = p$.
what is the probability that there is a string of k consecutive H?

$E_i =$ "coin tosses $i, i+1, \ldots, i+k-1$ are all H,

and $(i+k)$th toss is T".

why did we add this part? it makes E_i, E_j nearly independent!

$E_{n-k+1} =$ "coin tosses $n-k+1, \ldots, n$ are H"

no tosses left!

$P(E_i) = p^k(1-p)$

$P(E_{n-k-1}) = p^k$.

$$P(E_i \mid E_j) = \begin{cases} 0 & \text{if the sequences overlap!} \\ P(E_i) & \text{if sequences don't overlap.} \end{cases}$$

either way, it's pretty close to $P(E_i)$ which is small!

So $E_1, \ldots, E_{n-k+1}$ are nearly independent.

$X =$ "# of E_i that occur".

We want $X = X_\lambda$.

$\lambda = E[X] = (n-k)p^k(1-p) + p^k$

$P(X=0) = \exp\left(-(n-k)p^k(1-p) - p^k\right)\frac{\lambda^0}{0!}$

$\underline{p = 1/2 \text{ case:}}$

$$P(X=0) = \exp\left(\frac{-(n-k)+2}{2^{k+1}}\right) = \exp\left(\frac{-n}{2^{k+1}}\right)\exp\Big(\underbrace{\frac{k+2}{2^{k+1}}}_{\approx 0}\Big)$$

no strings of H of length k.

$$\approx \exp\left(\frac{-n}{2^{k+1}}\right)$$

If I want 10 consecutive H's, $k = 10$.

$$P(X=0) \approx \exp\Big(\frac{-n}{\underbrace{2^{11}}_{2048}}\Big)$$

don't get 10 consecutive H

$n = 1024$: $P(X=0) = e^{-1/2} = \frac{1}{\sqrt{e}} \approx 0.60$

$n = 2048$: $P(X=0) = e^{-1} = \frac{1}{e} = 0.36$

$n =$ ~~4096~~ ~~3072~~ 4096 $\quad P(X=0) = e^{-2} = \frac{1}{e^2} = 0.12$

5 seconds per coin toss.

		chance of success
1024 tosses →	1.42 hours	40%
2048 tosses →	2.84 hours	64%..
4096 tosses →	5.68 hours	88%..

How long it would take to get 10 H's in a row!