Lecture 22:

Previously, we looked at Bernoulli and Binomial random variables.

$B_{p,n} =$ # of successes in n trials where each trial can be successful with probability p.

$$P(B_{p,n} = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

We had examples like elections, communication systems.

Some properties of $B_{n,p}$:

$\star$ $\mathbb{E}[B_{n,p}] = np$ ✓ makes intuitive sense(?) but also check book or last time's notes for proof

$\star$ $\mathrm{Var}[B_{n,p}] = \mathbb{E}[(B_{n,p} - np)^2]$

$= np(1-p) \leftarrow p = \frac{1}{2}$ has the most variance.

$\star$ $$\frac{P(B_{n,p}=k)}{P(B_{n,p}=k-1)} = \frac{\frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}}{\frac{n!}{(n-k+1)!\,(k-1)!} p^{k-1} (1-p)^{n-k+1}} = \frac{(n-k+1)p}{k(1-p)}$$

which is > 1 when $k \le (n+1)p$. So $P(B_{n,p}=k)$ is increasing until $k > (n+1)p$

So $\max_k P(B_{n,p}=k)$ is attained when $k = \lfloor (n+1)p \rfloor$.

at the middle when $p = 1/2$.

useful for quickly computing $P(B_{n,p}=k)$ for various k.

Poisson Random-Variable:

Poisson r.v. is the r.v with prob. mass function:

$$p(i) = P(X=i) = e^{-\lambda}\frac{\lambda^i}{i!} \quad \leftarrow \text{whyyy???}$$

It comes from Binomial variable $B_{n,p}$ where n is large, p is small.

Setting $\lambda = np$ in Binomial variable: $\rightarrow p = \frac{\lambda}{n}$

$$P(X=i) = \frac{n!}{(n-i)!\,i!} p^i (1-p)^{n-i}$$

$$= \frac{n!}{(n-i)!\,i!}\left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \underbrace{\frac{n(n-1)\dots(n-i+1)}{n^i}}_{\text{approaches 1 for } i \ll n} \frac{\lambda^i}{i!} \frac{\overbrace{(1-\lambda/n)^n}^{\text{approaches } e^{-\lambda}}}{\underbrace{(1-\lambda/n)^i}_{\text{approaches 1 (for small } \lambda \text{) compared to } n}}$$

$$\boxed{P(X=i) \approx e^{-\lambda}\frac{\lambda^i}{i!}}$$

POISSON DISTRIBUTION.

← "An approximation of Binomial R.V. when n is large and p is small".

Ex: This models well:

- number of typos in a page.
- number of wrong telephone numbers dialed in a day.
- number of people who buy a specific item in a store per day.

  ↳ the idea is, lots of people go to that store, but each one buys the item with small probability.

Ex: Suppose # of typographical errors on a page is approximated by Poisson r.v. with $\lambda = 1/2$, what is the probability of having at least one error in a book's given page?

$$P(X=i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - e^{-\lambda} = 1 - e^{-1/2}$$