Lecture 21:

Last time: Bernoulli random variable

$$P(B_p = 0) = 1-p$$

$$P(B_p = 1) = p$$

Remark: $B_{p,1} = B_p$

n copies

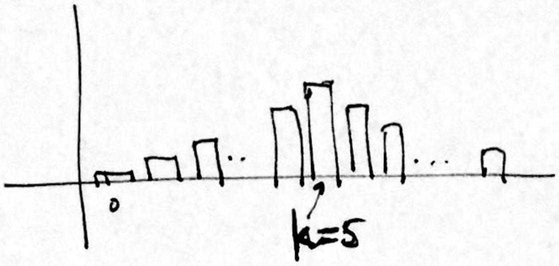
Binomial random variable: $B_p^{(1)} + B_p^{(2)} + \ldots + B_p^{(n)} = B_{p,n}$

$$P(B_{p,n} = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

eg: n experiments are performed, each one (independently) successful with probability p.
$B_{p,n}$ = # of successes.

Question/
Example: What is the most likely value of $B_{n,p}$?

We can plot for $p = \frac{1}{2}$, $n = 10$



also, if $p=0$ then 0 is most likely
if $p=1$ " 1 is most likely (100% likely)

But what about when $p \neq \frac{1}{2}$ and $0 < p < 1$?..

Look at:

$$\frac{P(X=k)}{P(X=k-1)} = \frac{\frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}}{\frac{n!}{(n-k+1)!(k-1)!} p^{k-1} (1-p)^{n-k+1}} = \frac{(n-k+1)p}{k(1-p)}$$

$(X = B_{n,p})$

So $P(X=k) \geq P(X=k-1)$ iff $(n-k+1)p \geq k(1-p)$
iff $k \leq (n+1)p$.

So $P(X=k)$ increases until after $k \leq (n+1)p$.
So largest value is at the largest integer $\leq (n+1)p$.

Ex: Communication system with n components.
Each component functions with probability p.
System works only when over half the components work.
Which is better: 3 components or 5 components?

5 component system: X = number of working components

$$P(X \geq 3) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + p^5$$

3-component system:

$$P(\underset{\text{different } X}{X} \geq 2) = \binom{3}{2} p^2 (1-p) + p^3$$

Comparing:

$$10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$$

reducing to:

$$3\underbrace{(p-1)^2}_{>0}(2p-1) > 0$$

ie. $p > \frac{1}{2}$.

So 5 is better if $p > 1/2$.

interpretation: If $p < \frac{1}{2}$ then you are more likely to get lucky with fewer tries.

Ex: (in homework) You have a country with 11 states. Candidate A is likely 60% to win each state. What is the probability that the candidate B (only other candidate) wins the election by winning at least 6 states?

- Expected value of binomial variable   n trials
probability p.

$$\mathbb{E}[X] = \sum_{i=0}^{n} \underbrace{i \binom{n}{i}} p^{i} (1-p)^{n-i}$$

$$\frac{i.n!}{(n-i)!\,i!} = \frac{i.n.(n-1)!}{(n-1-(i-1))!\,\cancel{i!}_{(i-1)!}} = n\binom{n-1}{i-1}$$

$$= \sum_{i=0}^{n} n\binom{n-1}{i-1} p^{i} (1-p)^{n-i}$$

$$= np \sum_{i=0}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

$$= np \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} (1-p)^{n-1-j}}_{1} \qquad j = i-1.$$

total probability ~~expectation~~ of
binomial variable with
$(n-1, p)$

$$= np.$$

would you have guessed this?

<u>Conclusion</u>:

$$\mathbb{E}[X] = np$$

$$\mathrm{Var}[X] = np(1-p)$$

proved using similar method.

Remark: We showed earlier that

$$\frac{P(B_{n,p} = k)}{P(B_{n,p} = k-1)} = \frac{(n-k+1)\,p}{k(1-p)}.$$

we can use this formula to compute $P(B_{n,p} = k)$ for each k starting at $k = 0$: $P(B_{n,p} = 0) = (1-p)^n$

Next up: What happens when $n \to \infty$ but p is small?

(Poisson)