Lecture 14:

Last time : independence

Def: E is independent from F if

$$P(EF) = P(E)\,P(F).$$

<u>equivalent definition:</u>

E is indep. of F if

$$P(E) = P(E \mid F).$$

<u>Ex:</u> E = card is $\spadesuit$, F = card is ace.

<u>Rmks</u> (1) different from mutually exclusive.

(2) It's a symmetric relation.. E is indep. of F iff F is indep. of E

(3) E is independent of F if and only if E is independent of F^c (proved later below)

<u>Ex:</u> two dice are thrown. (x_1, x_2)

F_4 = first die is 4 $\qquad (x_1 = 4)$

S_7 = sum total is 7. $\qquad (x_1 + x_2 = 7$

$$\underset{\displaystyle 1/6}{\underset{\shortparallel}{P(x_1 = 4 \mid x_1 + x_2 = 7)}} = \underset{\displaystyle 1/6}{\underset{\shortparallel}{P(x_1 = 4)}}$$

independent

<u>Ex:</u> Same setup: but now we are looking at $x_1 = 4$ and $x_1 + x_2 = 6$.

$$\underset{\displaystyle 1/5}{\underset{\shortparallel}{P(x_1 = 4 \mid x_1 + x_2 = 6)}} \neq \underset{\displaystyle 1/6}{\underset{\shortparallel}{P(x_1 = 4)}}$$

$\{(1,5),(2,4),(3,3),(\underline{\underline{4,2}}),(5,1)\}$

$x_1 \neq 6$ if $x_1 + x_2 = 6$, so if we know that $x_1 + x_2 = 6$, then probability that $x_1 = 4$ increases!

Proposition: E and F are independent if and only if E and F^c are independent.

proof: Assume E and F are independent

We know:

$$P(E) = P(EF) + P(EF^c)$$
$$= P(E)P(F) + P(EF^c)$$

so $$P(EF^c) = (1 - P(F))\,P(E)$$
$$= P(F^c)\,P(E).$$

Same argument, replacing F by F^c shows the other direction $\square$

Interestingly:

$$P(E \mid F) = P(E)$$

and $P(E \mid G) = P(E)$

does not imply necessarily that

$$P(E \mid FG) = P(G).$$

indeed:

$$P(x_1 + x_2 = 7 \mid x_1 = 3) = P(x_1 + x_2 = 7)$$

$$P(x_1 + x_2 = 7 \mid x_2 = 4) = P(x_1 + x_2 = 7)$$

but

$$\underbrace{P(x_1 + x_2 = 7 \mid x_1 = 3, x_2 = 4)}_{=1} \neq P(x_1 + x_2 = 7).$$

equivalently

$$P(E_i) = P(E_i \mid E_{i_1} E_{i_2} \dots E_{i_k})$$

any combination of others.

<u>Def:</u> Three events E, F, G are independent if:

$$P(EFG) = P(E)\,P(F)\,P(G)$$

$$P(EF) = P(E)P(F)$$

$$P(EG) = P(E)\,P(G)$$

and $P(FG) = P(F)\,P(G)$

knowing any combination of the others does not change the probability that E_i is true.

Many events $E_1, E_2, \dots, E_n$ are independent

if $P(E_{i_1} E_{i_2} \dots E_{i_k}) = P(E_{i_1}) \dots P(E_{i_k})$

for every subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$.

Example: An infinite sequence of trials is performed.
Each trial results in success with prob. p.
and failure with prob $1-p$.
What is the probability that:

(1) At least 1 success occurs in n trials

(2) Exactly k successes in n trials

(3) All trials are successes.

Solution:

Let $E_i =$ "ith trial is success".

(1) $$P((E_1 \cup E_2 \cup \dots \cup E_n)^{c\,c}) = 1 - P(E_1^c \cap E_2^c \dots \cap E_n^c)$$
$$= 1 - P(E_1^c)P(E_2^c)\dots P(E_n^c)$$
$$= 1 - (1-p)(1-p)\cdots(1-p).$$

(2) Probability $$P(\underbrace{E_{i_1} E_{i_2} \dots E_{i_k}}_{\text{successful ones}} \underbrace{E_{j_1}^c \dots E_{j_{n-k}}^c}_{\text{not successful ones}})$$

$$= P(E_{i_1})\dots P(E_{i_k}) P(E_{j_1}^c)\dots P(E_{j_{n-k}}^c)$$

$$= p^k (1-p)^{n-k}.$$

but we can choose the k successful ones in $\binom{n}{k}$ ways
so probability that exactly k are successful:

$$= \binom{n}{k} p^k (1-p)^{n-k}.$$

(3) $$P(E_1 E_2 E_3 \dots) = \prod_{i=1}^{\infty} P(E_i) = \lim_{n\to\infty} p^n = \begin{cases} 1 & \text{if } p = 1 \\ 0 & \text{if } p \leq 1 \end{cases}$$

Ex: Problem of the 5 before 7:

I keep rolling a pair of dice. What is the probability that sum will be 5 before it ever is 7.

solution: • On any single trial

$$P(x_1+x_2=5)=\frac{4}{36}$$

$$P(x_1+x_2=7)=\frac{6}{36}=\frac{1}{6}$$

$E_n=$ "we didn't get 5 or 7 for the first $n-1$ times, but got 5 on the nth try".

we are looking for $P\left(\bigcup_{n=1}^{\infty} E_n\right)=\sum P(E_n)$

$$P(E_n)=\underbrace{\left(1-\frac{10}{36}\right)^{n-1}}_{\text{don't get 5 or 7}}\cdot\underbrace{\frac{4}{36}}_{\text{get 5 at the end}}=\left(\frac{13}{18}\right)^{n-1}\cdot\frac{1}{9}$$

$$\sum_{n=1}^{\infty}P(E_n)=\frac{1}{9}\sum_{n=1}^{\infty}\cancel{\tfrac{1}{9}}\left(\frac{13}{18}\right)^{n-1}=\frac{1}{9}\,\frac{1}{1-\frac{13}{18}}=\frac{2}{5}.$$

other solution (very cool) $E=$ "we get a 5 before a 7 (eventually)".

$F=$ "first try is 5" $G=$ "first try is 7" $H=$ "first try is neither"

$$P(E)=\underbrace{P(E|F)}_{1}P(F)+\underbrace{P(E|G)}_{0}P(G)+\underbrace{P(E|H)}_{=P(E)}P(H)$$

$$=P(F)+P(E)P(H)=\frac{1}{9}+P(E)\frac{13}{18}$$

THEN SOLVE FOR $P(E)$.