Lecture 9: Hard problems:

(In the lecture, we did the problems at end of notes for lecture 7 ~~and then~~ ...)

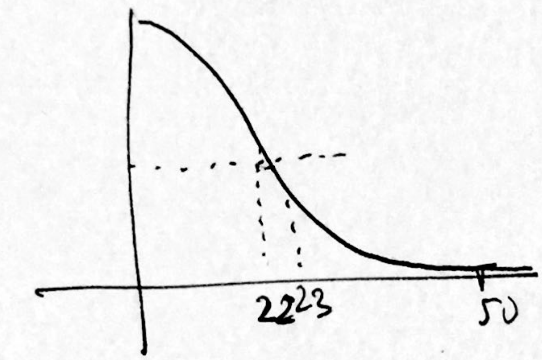
More examples of uniform distribution problems.

Ex1: n people are in a room. What is the probability that no two of them have the same birthday. How many people $n = ?$ makes this probability $> 50\%$?

Solution:

$$P = \frac{\#\text{ of ways no two birthdays coincide}}{\#\text{ of ways } n \text{ birthdays can be.}} = \frac{365 \cdot 364 \cdots (365-n+1)}{365^n}$$

calculating with computer:



when $n = 23$

$p = 0.4927$

when $n = 50$

$p = 0.029.$

Ex2 (Hat) matching problem.

n people throw their hats, hats get randomly mixed up. What is the probability that nobody gets their own hat?

Solution: $P(\text{everyone gets their own hat}) = \frac{1}{n!}$

because it's one permutation $(1\,2\,3 \ldots n)$ among $n!$ possible permutations of hats.

Let E_i = the event that ith person gets their own hat

$= \{(a_1, \ldots, a_n) \mid a_i = i\} \subset$ "Permutations" $= S$

$|S| = n!$

In order to compute $P(\text{nobody gets their own hat})$

$$= P(S \setminus (E_1 \cup E_2 \cup \ldots \cup E_n))$$

Let us compute $P(E_1 \cup E_2 \cup \ldots \cup E_n)$.

$$P(E_1 \cup \ldots \cup E_n) = \sum_i P(E_i) - \underbrace{\sum_{i_1<i_2} P(E_{i_1} E_{i_2})}_{\binom{n}{2} \text{ terms}} + \underbrace{\sum_{i_1<i_2<i_3} P(E_{i_1} E_{i_2} E_{i_3})}_{\binom{n}{3} \text{ terms}} - \ldots$$

Can we compute?

$$P(E_{i_1} E_{i_2} E_{i_3} \ldots E_{i_k})$$

Yes, it's as follows the hats for $i_1, \ldots, i_k$ are fixed, the others can be permuted any way we want.

$$P(E_{i_1} \ldots E_{i_k}) = \frac{(n-k)!}{n!}$$

$(n-k)!$ ← # of ways to permute remaining hats.

$n!$ ← all hat permutations.

$$\sum_{i_1 < \ldots < i_k} P(E_{i_1} \ldots E_{i_k}) = \frac{\cancel{n!}}{\cancel{(n-k)!}\,k!} \frac{\cancel{(n-k)!}}{\cancel{n!}} = \frac{1}{k!}$$

So

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots + (-1)^{n+1}\frac{1}{n!} \approx \frac{1}{e} = e^{-1} \text{ for } n \gg 1$$

↑ Taylor series

remark: the probability doesn't approach 1 as $n \to \infty$!

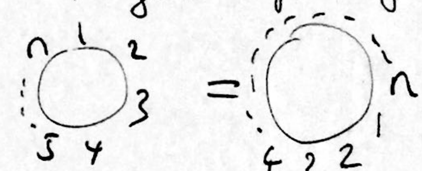
Ex 3: 10 couples (20 people) are ~~sat~~ (individually) randomly at ~~the~~ a table that is round.
What is the probability that no husband sits next to his wife?

Solution: We will solve this by computing

$$1 - P(\text{some couple end up sitting next to each other}).$$

How many ways can we sit n people at a round table anyway?

$$\frac{n!}{n} \leftarrow$$ divide by n because rotating everyone doesn't change seating arrangement

ie 

Let E_i be the event that the ith couple sit next to each other. We want $1 - P\left(\bigcup_{i=1}^{10} E_i\right)$

$$P\left(\bigcup_{i=1}^{10} E_i\right) = \sum_{i=1}^{10} P(E_i) - \ldots + (-1)^{k+1} \sum_{i_1<i_2<\ldots<i_k} P(E_{i_1} \ldots E_{i_k}) + \ldots - P(E_1 \ldots E_{10})$$

$$P(E_{i_1} \ldots E_{i_k}) = \frac{\frac{(20-k)!}{(20-k)}\, 2^k}{19!}$$

→ # of ways to seat 20 people at round table

# of ways to seat 20 people with k couples sitting next to each other.

Think of each couple sitting next to each other as a single item. There are k couples next to each other and $20-2k$ persons besides those. So $20-k$ entities. They can be seated $\frac{(20-k)!}{(20-k)}$ ways but we can also switch the husband and wife for each of the k couples (2^k)