Lecture 7: We were looking at consequences of the 3 axioms of probability:

(1) $1 \geq P(E) \geq 0$

(2) $P(S) = 1$

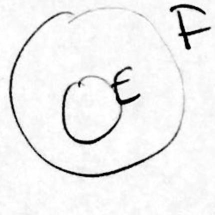
(3) If $E_1, E_2, \ldots$ are some mutually exclusive events (ie. $E_i E_j = \emptyset \ \forall i,j$)
then:
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Consequences:

$$\left.\begin{array}{l}(1)\ P(\emptyset) = 0 \\ (2)\ P(E^c) = 1 - P(E)\end{array}\right\}\text{ proven last time}$$

(3) If $E \subset F$, then $P(E) \leq P(F)$.

proof:

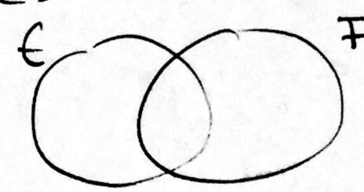


$$F = E \cup (\underbrace{E^c \cap F}_{F \setminus E})$$

So $$P(F) = P(E) + \underbrace{P(E^c \cap F)}_{\geq 0 \text{ by 1st axiom}}$$

So $P(F) \geq P(E)$

(4) $P(E \cup F) = P(E) + P(F) - P(EF)$



$$\left.\begin{array}{l} P(E \cup F) = P(F \setminus E) + P(E \setminus F) + P(EF) \\ P(E) = P(E \setminus F) + P(EF) \\ P(F) = P(F \setminus E) + P(EF) \end{array}\right\}\text{ combine}$$

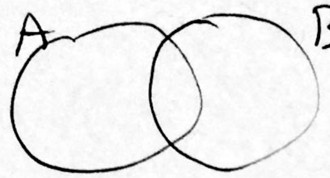
Ex: I am reading two books.

P(I like 1st book) = 0.5

P(I like 2nd book) = 0.4

P(I like both books) = 0.3

P(I don't like either) = ?



solution:

$P(A) = 0.5$

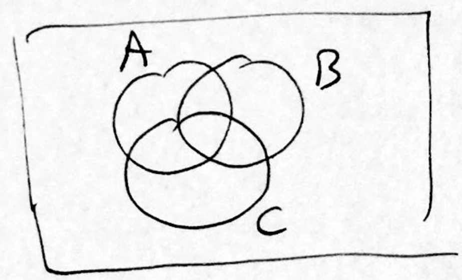
$P(B) = 0.4$

$P(AB) = 0.3$

$$P(A \cup B) = P(A) + P(B) - P(AB)$$
$$= 0.5 + 0.4 - 0.3 = 0.6$$

3-way version:

$P(A \cup B \cup C)$



$$= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC).$$

n-way version (inclusion-exclusion principle)

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum P(E_i) - \sum P(E_{i_1} E_{i_2})$$
$$+ \sum P(E_{i_1} E_{i_2} E_{i_3})$$
$$- \vdots$$
$$+ (-1)^{n+1} P(E_1 E_2 \dots E_n).$$

summation over all pairs unordered $\{i_1, i_2\}$ $\binom{n}{2}$ of them.

Problems where every element of the sample space is equally likely (uniform distribution)

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$$

Ex 1: Two dice are rolled, Prob. that sum is 7?

$$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$p = \frac{6}{36} = \frac{1}{6}$$

Ex2: Committee of 5 is randomly selected from 6 men and 9 women. What is the probability that 3 men and two women are selected?

$$\frac{\binom{6}{3}\cdot\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001} \leftarrow \frac{\cancel{\text{prob}}\ \#\text{ of ways}}{\#\text{ of ways}}$$

Ex3: n balls, k are chosen. One ball is special, what is the probability that the special ball is chosen?

$$\frac{\text{number of ways to choose } k \text{ balls including special ball}}{\binom{n}{k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

Ex4: deck of 52 cards dealt to 4 players. Prob that player 1 gets all spades $(\spadesuit)$.

- Number of dealings where player 1 gets all spades?
- probability that player 1 gets all spades? $\leftarrow$ easier to do directly

$$\frac{1}{\binom{52}{13}}$$ because player 1 gets 13 random cards only one hand is all spades.

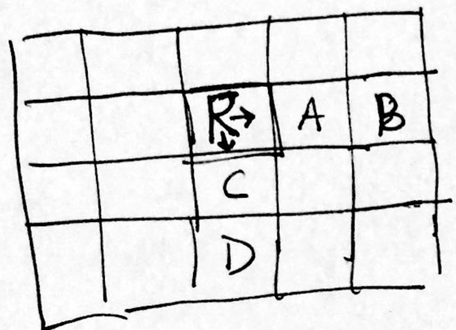
Ex5: Same but "one player gets all spades:

A: $\dfrac{4}{\binom{52}{13}}$ because it's P(player 1 gets all spades)
+ P(player 2 gets all spades)
+ ...

(mutually exclusive events)

Probability as measure of belief:

Say I am writing software for a robot in a grid.



- Robot "thinks" there is 30% chance of finding something to clean at location A, 20% chance at location B.
- 40% at location C, 10% chance at location D.

Should the robot go right or down? (assuming robot wants to clean up at least one square)

We want to compute the prob. that there is something at location A or B. $1 - \underbrace{0.7 \cdot 0.8}_{\substack{\text{nothing} \\ \text{at A or B.}}} = 0.44$.

C or D: $1 - 0.6 \cdot 0.9 = 0.46$