Lecture 6:

Last time: • Sample space S → eg $S = \{(H,H), (H,T), (T,H), (T,T)\}$.

• Event $E \subset S$

probability: P : "events" $\to \mathbb{R}$ numbers.

A bit about subsets: $E_1, E_2 \subset S$ (events)

* You can intersect them:

$E_1E_2 = E_1 \cap E_2$

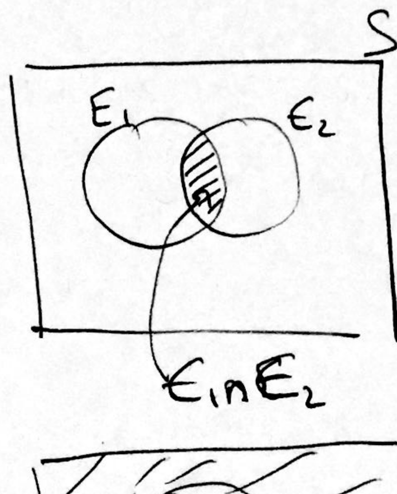
this is a new event (<u>both</u> events occurring)

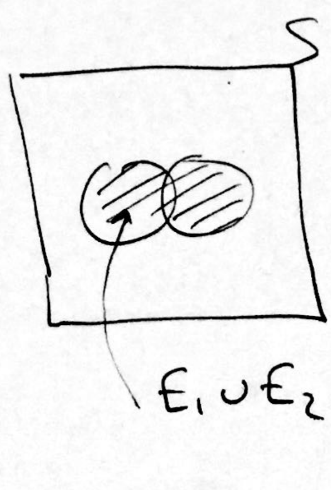
* $E_1 \cup E_2$ (<u>either</u> event occurring)

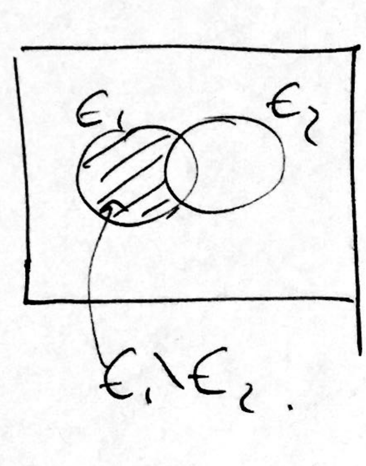
the union

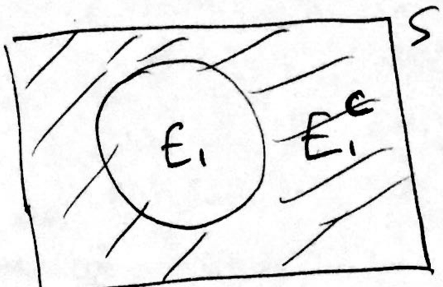
* $E_1^c = S \setminus E_1$ ← difference.

* $E_1 \setminus E_2 = E_1 \cap E_2^c$ (E_1 occurring but E_2 not occurring)









- commutative laws: $E \cup F = F \cup E$

$$EF = FE$$

- associative laws: $(E \cup F) \cup G = E \cup (F \cup G)$
- distributive laws $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

$$G \cap (E \cup F) = (G \cap E) \cup (G \cap F).$$

($\cap$ is like multiplication and $\cup$ is like addition).

<u>Fun fact:</u> $(E_1 \cup E_2)^c = E_1^c \cap E_2^c$

$$\left(\bigcup_{i=1}^{\infty} E_i\right)^c = \bigcap_{i=1}^{\infty} E_i^c.$$

similar for intersection's complement.

<u>Axioms of probability:</u> S sample space, with some events P satisfies the following.

① $0 \leq P(E) \leq 1$

② $P(S) = 1$

I called it independent also.

③ For $E_1, E_2, \ldots$ mutually <u>exclusive</u> events,
(ie $E_i \cap E_j = \emptyset$ for all i, j)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Consequences of the axioms:

(1) $P(\emptyset) = 0$. $\quad$ because $S \cap \emptyset = \emptyset$.

because: $P(S) = P(S \cup \emptyset)$

$$= P(S) + P(\emptyset)$$

so $P(\emptyset) = 0$.

(2) $P(E^c) = 1 - P(E)$

proof:

$$1 \overset{\text{2nd axiom}}{=} P(S) = P(\underbrace{E \cup E^c}_{S}) \overset{\text{3rd axiom}}{=} P(E) + P(E^c)$$

so $P(E^c) = 1 - P(E)$.

(3) If $E \subset F$, then $P(E) \leq P(F)$.

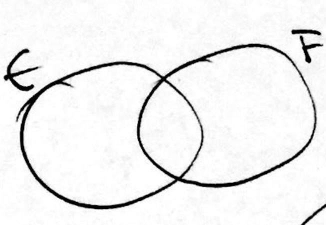
proof: 

$F = E \cup (E^c \cap F)$.

$$P(F) = P(E \cup (E^c \cap F)) = P(E) + \underbrace{P(E^c \cap F)}_{\geq 0 \text{ by 1st axiom}}$$

So $P(F) \geq P(E)$

(4) $P(E \cup F) = P(E) + P(F) - P(EF)$.



$$\begin{cases} P(E \cup F) = P(F \setminus E) + P(E \setminus F) + P(EF) \\ P(E) = P(E \setminus F) + P(EF) \\ P(F) = P(F \setminus E) + P(EF) \end{cases}$$

Combine these three.