Lecture 4:

Last week: counting, $\binom{n}{r}$

Binomial theorem. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Finally that antennas problem from 1st lecture:

n antennas in a line. m broken $(m \leq n)$

1 0 1 1 0 ✓

1 $\underline{\underline{0\,0}}$ 1 1 ✗

If two consecutive are broken, then system is broken.

How many ways can the system be working?.

Solution: clever idea: Imagine 1's are placed

$$\_\,1\,\_\,1\,\_\,1\,\_ \;\cdots\; \_\,1\,\_$$

$n-m+1$ spots.

we can put the 0's in these spots. But each spot can have at most one zero.

So we must choose m out of $\underbrace{n-m+1}_{\text{one more than \# of 1's}}$ spots.

answer $\binom{n-m+1}{m}$.

<u>Ex</u>: How many ways can I split:

$$n = x_1 + x_2 + \dots + x_r$$

e.g. I have n identical candies. r distinct kids... (each kid gets at least one candy)

where $x_i > 0$, $x_i \in \mathbb{Z}$ integers.

<u>soln</u>: $r = 3$
$n = 5$

$$\left.\begin{array}{ccc} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{array}\right\} \text{6 ways.}$$

clever: place n 1's in a line.

$n-1$ spots.

$$\underbrace{1 \,\vdots\, 1}_{x_1} \Big| \underbrace{1 \,\vdots \cdots \vdots\, 1}_{x_2} \Big| \underbrace{1 \,\vdots\, 1}_{x_3}$$

place separators.. $r-1$ chosen.

At most one separator per spot. So we must choose

$$\binom{n-1}{r-1}.$$

<u>Ex</u>: How many ways can I split:

$$n = x_1 + x_2 + \dots + x_r$$

where each $x_i \geq 0$. ($x_i \in \mathbb{Z}$ integers.)

<u>Solution</u>: more cleverness: Splitting n as a sum of r non-negative numbers is the same as splitting $n+r$ as a sum of positive numbers.

$$n + r = y_1 + \dots + y_r.$$

And the corresponding $x_i = y_i - 1$.

So final answer:

$$\binom{n+r-1}{r-1}.$$

Ex: How many terms are there in the expansion of

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1 \ldots n_r} x_1^{n_1} \cdots x_r^{n_r}?$$

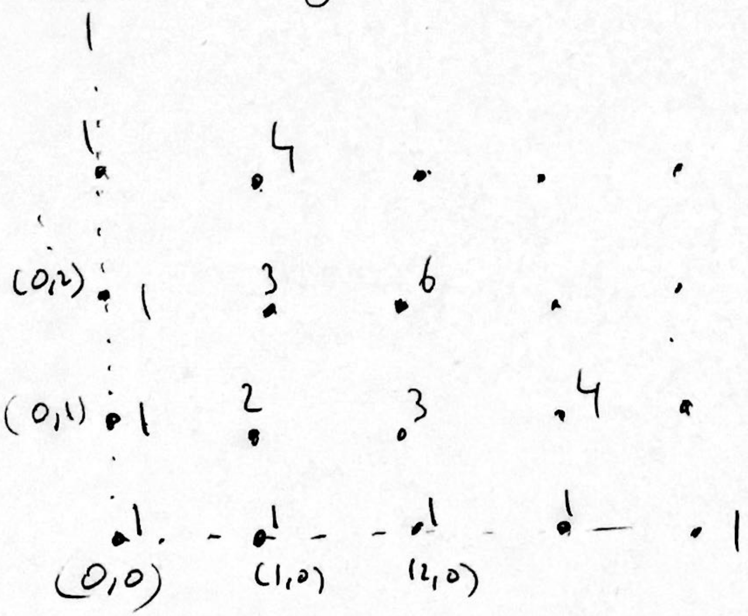
(equivalently: how many coefficients would I need to give you in order to specify a degree n polynomial in r variables?)

We want to see how many ways we can split.

$$n_1 + n_2 + \cdots + n_r = n.$$

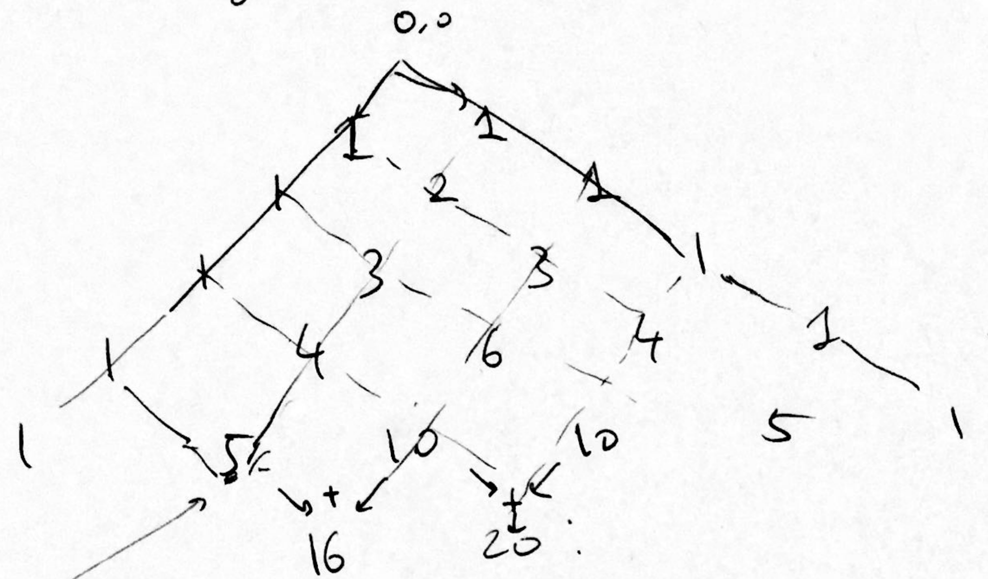
Answer is $\binom{n+r-1}{r-1}$.

One more thing about counting: Pascal's triangle



Start at $(0,0)$
At each step, we can go right or up. How many ways can I get to (m,n)?
it's $\binom{m+n}{n}$ because I must take $m+n$ steps, but exactly m of them must be to the right.

write down at each spot, how many ways I can get there.



If I wat to get to (m,n),
I can get there from the left or from below.
so computing each:

$$\binom{m+n}{m} = \binom{m+n-1}{m-1} + \binom{m+n-1}{m}$$

we already knew this formula.

$$(x+y)^5 = \underline{1}x^5 + \underline{5}x^4y + \underline{10}x^3y^2 + \underline{10}x^2y^3 + \underline{5}xy^4 + 1y^5.$$