Lecture 3: Last time: - choose function $\binom{n}{k}$

= "number of ways to choose k items (unordered) from n distinct items".

- $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Lemma: $\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{r} + \dots + \binom{n}{n} = 2^n$

proof: 2^n and $\sum_{r=0}^{n} \binom{n}{r}$ are two ways of counting the number of subsets of the set $\{1,2,\dots,n\}$.

- Each element is either in the subset or out, so 2 choices for 1st element 2 choices for second element ...

$$\underbrace{2 \cdot 2 \cdot 2 \dots \cdot 2}_{n} = 2^n \text{ subsets}$$

- On the other hand, we can count the number of subsets with 0 elements, 1 element, 2 elements, ..., n elements

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}.$$

There is another way to prove this:

<u>Binomial theorem</u>:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

<u>proofs</u>:

- combinatorial proof

$$(x+y)(x+y)(x+y)\cdots\cdots(x+y)$$

to get a term $x^k y^{n-k}$, you need to multiply the x from k of the $(x+y)$'s with the y's from $n-k$ of the $(x+y)$'s. how many ways to pick k of the n $(x+y)$'s. $\binom{n}{k}$. So there are $\binom{n}{k}$ many $x^k y^{n-k}$ in the expansion.

- proof by induction.

<u>Base case</u>: $n=1$

$$(x+y)^1 = \binom{1}{0} x + \binom{1}{1} y \quad \checkmark$$

<u>Induction step</u>:

Assume $(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}$.

$$(x+y)^{n-1}(x+y) = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} (x+y)$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}$$

put in $s=k+1$ and $k=s-1$ in 1st sum:

$$= \sum_{s=1}^{n} \binom{n-1}{s-1} x^s y^{n-s} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}$$

$$= \sum_{k=1}^{n-1} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^k y^{n-k} + x^n + y^n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k} x^k y^{n-k} + x^n + y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

1, 5, 10, 10, 5, 1.

<u>Ex</u>: $(x+y)^5 = x^5 + \binom{5}{1} x^4 y^1 + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} x y^4 + \binom{5}{5} y^5$

<u>Back to the Antenna problem from first lecture:</u>

You have n antennas in a line, m are broken
if two antennas which are consecutive are broken, then the system is broken.
How many ways can the system be working?

We want sequences of m 0's
$n-m$ 1's

with no two 0's next to each other.

<u>Solution</u>: Imagine we have the 1's:

$$\_\,1\,\_\,1\,\_\,1\,\_\,1 \ldots \_\,1$$

we want to place 0's in these spots. We can have at most one 0 per spot. So we need to choose m spots among: $n-m+1$

$$\binom{n-m+1}{m}.$$

Multinomial coefficients:

Choose function shows how many ways to split n items into 2 groups of fixed sizes r and $n-r$ (r chosen and $n-r$ not chosen)

What if we want to split into more groups?

Ex: How many ways can we split 10 items into a group of 2, group of 5, and group of 3.

Soln: There are $10!$ ways to order them.

eg:

$$\underbrace{1\ 3}_{\text{1st group}}\ \underbrace{5\ 2\ 4\ 6\ 7}_{\text{2nd group}}\ \underbrace{9\ 8\ 10}_{\text{3rd group.}}$$

But we are over-counting since each group should be un-ordered. No problem, divide by # of orderings within each group:

$$\text{answer} = \frac{10!}{2!\,5!\,3!}$$

In general: $n_1, n_2, \ldots, n_r$ sizes of groups

$$n = n_1 + n_2 + \cdots + n_r$$

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \ldots n_r!}$$

Multinomial coefficients version of binomial theorem:

$$(x_1+x_2+\cdots+x_r)^n = ?$$

we want to expand.

- each term will have total degree n.

$$x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

$n_1+n_2+\cdots+n_r = r$.

Counting # of each term is:

- Same as splitting the ~~terms~~ $(x_1+\cdots+x_r)$'s into groups.

$$(x_1+x_2+\cdots+x_r)^n = \sum \binom{n}{n_1,n_2,\ldots,n_r} x_1^{n_1}\cdots x_r^{n_r}.$$