Lecture 2:

Last time: counting & permutations.
There are $n!$ ways to order n distinct items.
Ex: Say I have 100 job candidates and 8 distinct positions in my company (VP of engineering, C.O.O., VP of marketing... etc.)
How many ways can I fill these positions?

solution: $\underbrace{100}_{\text{choices for first}} \cdot \underbrace{99}_{\text{choices for second position}} \cdot 98 \cdot \ldots \cdot 93$

In general, if I want to pick for specific positions, r people among n.

$$n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Ex: How many different words can you make by reordering the letters in PEPPER.

soln: If all the letters were different, we would have: $6!$ orderings.
But we are over-counting because, for example

$$P_1 E_1 P_2 P_3 E_2 R_1 \text{ and } P_1 E_2 P_2 P_3 E_1 R_1$$

are the same word.

How much are we over-counting?

We are counting each word as many times as we can shuffle the letters around without changing the word.

PEPPER

i.e shuffle the E's amongst themselves 2 ways.

shuffle the P's amongst themselves $3! = 6$ ways.

So we are counting each word $2 \cdot 6 = 12$ ways.

Answer: $\frac{6!}{12} = \frac{\cancel{6} \cdot 5 \cdot 4 \cdot 3 \cdot \cancel{2}}{\cancel{12}} = 60$.

<u>Combinations</u>

<u>Ex:</u> I am invited to a party by the Queen of England and I can bring three friends with me. I have 100 friends (real friends). How many different ways can I choose people to come with me?

In general, how do I choose r out of n?

The difference is that the people are not ordered, they are not for fixed positions, so choosing

Johnny, Ashley, Sun

is same as choosing

Ashley, Sun, Johnny.

I would normally have

$$n \cdot (n-1) \cdots (n-r+1)$$

if positions were fixed. But I am over-counting by $r!$ since reordering doesn't matter. So answer is:

$$\frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} =: \binom{n}{r}$$

definition

"n choose r".

$$\binom{n}{r} = \text{"number of ways to choose } r \text{ among } n \text{ distinct items".}$$

Ex: I have to choose 6 numbers between 1 and 49 for the lotto. How many ways?

$$\binom{49}{6} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{6!} = 13{,}983{,}816$$

Properties of choose function

- $\binom{n}{0} = 1$ ← only one way to choose nothing.

- $\binom{n}{n} = 1$ ← only one way to choose everything

- $\binom{n}{r} = \binom{n}{n-r}$ ← choose r things to be included, or choose $n-r$ to be excluded. same thing.

Say I am choosing r things from n.
I can either choose the first thing or not.
If I choose the first thing, then I must choose $r-1$ things amongst $n-1$.
If I don't choose the first, then I must choose r things amongst $n-1$. So:

- $$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

- $$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

I am counting subsets of $\{1, 2, \ldots, n\}$. There are 2^n subsets. (each element in or out)
There are $\binom{n}{0}$ subsets with 0 elements, $\binom{n}{1}$ with 1 element, $\binom{n}{i}$ with i elements....