HOMEWORK 1

Due Thursday, April 27, at the beginning of discussion

1. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack (i.e. one of the cards is an ace, and the other is either a ten, jack, queen or king)?

2. A bridge hand consists of 13 cards randomly dealt from a regular deck of 52 cards. What is the probability that a bridge hand does not contain any card of at least one suit?

3. An urn contains 3 red and 7 black balls. Players A and B randomly draw balls from the urn consecutively until a red ball is selected. Assuming that A draws the first ball, and that the balls are drawn without replacement, find the probability that A selects the red ball.

4. An urn contains n white and m black balls, where n and m are positive integers.
   (a) If the two balls are randomly drawn, what is the probability that they are of the same color? *This is sampling "without replacement".*
   (b) If one ball is randomly drawn from the urn, its color noted, and then replaced back into the urn, after which the second ball is randomly drawn, what is the probability that they are of the same color? *This is sampling "with replacement".*

5. A woman has n keys, of which, one will open her door.
   (a) If she tries her keys at random, keeping aside those that don't work, what is the probability that she will open the door on her kth try?
   (b) What if she doesn't keep aside previously tried keys?

6. In class, we learnt the inclusion-exclusion identity:

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{n} E_i\right) &= \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P\left(E_{i_1} \cap E_{i_2}\right) + \sum_{i_1<i_2<i_3} P\left(E_{i_1} \cap E_{i_2} \cap E_{i_3}\right) + \ldots \\
&+ (-1)^{r-1} \sum_{i_1<i_2<\ldots<i_r} P(E_{i_1} \cap E_{i_2} \cap \ldots \cup E_{i_r}) + \ldots \\
&\ldots + (-1)^{n-1} P(E_1 \cap E_2 \cap \ldots \cap E_n)
\end{aligned}
$$

Give a proof of the inclusion-exclusion identity using mathematical induction. (You may assume the case for $n = 2$ to be true, since we proved it in class.)

7. Prove Bon-Ferroni's inequality: For any two events, E and F, of a sample space, S,

$$P(E \cap F) \geq P(E) + P(F) - 1.$$

8. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be
   (a) no complete pair?
   (b) exactly 1 complete pair?
9. A pair of fair dice are rolled. Using the definition of conditional probability, find
   (a) the conditional probability that at least one lands on 6, given that the dice land on different numbers.
   (b) the conditional probability that the first one lands on 6, given that the sum of the dice is 8.
   (c) the conditional probability that at least one of the two dice lands on 6, given that the sum of the dice is 8.