Lecture 28:

Last time: we stated the first isomorphism theorem. We'll prove it today and talk about some examples.

Recap: • $H \leq G$ normal if $\forall g \in G$ $gHg^{-1} \subset H$.

• If H is normal, we can have a quotient group $G/H = \{gH \mid g \in G\}$ = set of left cosets with operation $g_1H\, g_2H = g_1g_2H$

• $|G/H| = |G|/|H|$

Theorem: (1st isomorphism theorem) Let $\varphi : G \longrightarrow K$ be a group homomorphism. The map

$$\tilde{\varphi} : G/\ker\varphi \longrightarrow \operatorname{Im}\varphi$$

given by $\tilde{\varphi}(g\ker\varphi) = \varphi(g)$ is an isomorphism.

proof: • We first check that $\tilde{\varphi}$ is well-defined. Assume $g_1\ker\varphi = g_2\ker\varphi$, then $g_2 = g_1k$ for some $k \in \ker\varphi$. Then $\tilde{\varphi}(g_2\ker\varphi) = \varphi(g_2) = \varphi(g_1k) = \varphi(g_1)\underbrace{\varphi(k)}_{e} = \tilde{\varphi}(g_1\ker\varphi)$.

• $\tilde{\varphi}$ is a homomorphism since $\tilde{\varphi}(g_1\ker\varphi \cdot g_2\ker\varphi)$
$= \tilde{\varphi}(g_1g_2\ker\varphi) = \varphi(g_1g_2)$
$= \varphi(g_1)\varphi(g_2) = \tilde{\varphi}(g_1\ker\varphi)\tilde{\varphi}(g_2\ker\varphi)$

- $\tilde{\varphi}$ is surjective by definition (since the target of $\tilde{\varphi}$ is image of φ)

- $\tilde{\varphi}$ is injective since

$$\operatorname{Ker}\tilde{\varphi} = \{g_1 \operatorname{Ker}\varphi \mid \tilde{\varphi}(g_1 \operatorname{Ker}\varphi) = e\}$$
$$= \{g_1 \operatorname{Ker}\varphi \mid \varphi(g_1) = e\} = \{\operatorname{Ker}\varphi\}.$$

$\square$

<u>Examples</u>: (1) Let $\varphi : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$

be given by $\varphi(\bar{a}) = 4\bar{a}$.

$\operatorname{Ker}\varphi = \{\bar{a} \in \mathbb{Z}/8\mathbb{Z} \mid 4\bar{a} = \bar{0}\} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} = \langle \bar{2} \rangle$

$\operatorname{Im}\varphi = \{\bar{0}, \bar{4}\}$

So $\quad (\mathbb{Z}/8\mathbb{Z}) / \langle \bar{2} \rangle \cong \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\operatorname{Im}\varphi} = \{\bar{0}, \bar{4}\}.$

(2) $GL_n \xrightarrow{\det_n} \mathbb{R}^\times$.

$\ker \det_n = SL_n$. $\quad \operatorname{Im} \det_n = \mathbb{R}^\times$.

$GL_n / SL_n \cong \mathbb{R}^\times$

automatically normal because it's the kernel.

(3) $S_n \xrightarrow{sgn} \{+1, -1\}$

← multiplicative group

$$sgn(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

$\ker sgn = A_n$ ← all even permutations.

So $S_n / A_n \cong \{+1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$

again automatically normal

this comes from the isomorphism theorem.

<u>Remark</u>: A subgroup $H \subset G$ is normal if and only if it is the kernel of a homomorphism $\varphi : G \to K$ for some group K.

(normal subgroups are kernels).

(4) Let H, K be two groups:
consider the projection $\pi_1 : H \times K \to H$
$\pi_1(h,k) = h$

$\ker \pi_1 = \{(h,k) \mid h = e\}$
$= \{e\} \times K$

So $H \times K / \{e\} \times K \cong H = \operatorname{Im} \pi_1$.

Similarly: $H \times K / H \times \{e\} \cong K$.