Lecture 24: Homomorphisms

We already know what a homomorphism is, but we were never interested in homomorphisms which were not isomorphisms.

Def: A map $f: G_1 \to G_2$ is a homomorphism if,

$$\forall g, h \in G, \quad f(g \cdot h) = f(g) f(h).$$

Examples:

(1) $\varphi: \mathbb{Z} \to \mathbb{Z}$

$\varphi(k) = ak$ for fixed a.

$$\begin{aligned} \varphi(k_1 + k_2) &= a(k_1 + k_2) \\ &= ak_1 + ak_2 \\ &= \varphi(k_1) + \varphi(k_2) \end{aligned}$$

- φ is injective unless $a = 0$
- φ is bijective only when $a = 1$ or -1.

(2) $\varphi: \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

$\varphi(k) = \overline{k}$.

this was the definition of $+$ on $\mathbb{Z}/n\mathbb{Z}$.

$$\varphi(k_1 + k_2) = \overline{k_1 + k_2} = \overline{k_1} + \overline{k_2} = \varphi(k_1) + \varphi(k_2)$$

φ is surjective but not injective. (eg $\varphi(0) = \varphi(n)$)

(3) Consider the group $(\{1,-1\}, \underset{\text{multiplication}}{\cdot})$

$$\text{sign}: S_n \longrightarrow \{1,-1\}$$

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

is a homomorphism.

$$\varphi(\underbrace{\sigma_{odd} \cdot \sigma_{odd}}_{\substack{\text{even} \\ \shortparallel \\ 1}}) = (-1)\cdot(-1) = \varphi(\sigma_{odd})\,\varphi(\sigma_{odd})$$

similarly for other cases.

(4) determinant:

$$\det: GL_n(\mathbb{R}) \longrightarrow (\mathbb{R}\setminus\{0\})$$

$$\det(A \cdot B) = \det A \det B.$$

so det is a homomorphism.

(5) Trivial homomorphism:

$$\varphi: G_1 \to G_2$$

$$\varphi(g) = e \text{ for all } g \in G_1.$$

$$\varphi(g_1 g_2) = e = \underbrace{\varphi(g_1)}_{e}\,\underbrace{\varphi(g_2)}_{e}.$$

(6) Projections $\pi_1 : G_1 \times G_2 \longrightarrow G$

$$\pi_1(g_1, g_2) = g_1$$

similarly π_2.

We know from before:

If $\varphi: G_1 \to G_2$ is a homomorphism, then

- $\forall g \in G_1 \quad \varphi(g)^{-1} = \varphi(g^{-1})$

- $\varphi(e_{G_1}) = e_{G_2}$

<u>Lemma</u>: If $\varphi: G \to K$ is a homomorphism then:

(1) $\forall H \leq G$ subgroup, $\varphi(H)$ is a subgroup of K

(2) $\forall H \leq K$ subgroup, $\varphi^{-1}(H) \leq G$ is a subgroup

<u>proof</u>: we did (1) before (do it as exercise if you don't remember.)

(2) We need to show that $\varphi^{-1}(H)$ is closed under multiplication and inverses, and contains e.

$\{x \in G \mid \varphi(x) \in H\}$. "inverse image of H"

- $e \in \varphi^{-1}(H)$ since $\varphi(e) = e$.
- If $g_1, g_2 \in \varphi^{-1}(H)$, then $\varphi(g_1) \in H$ and $\varphi(g_2) \in H$.

  so $\varphi(g_1) \cdot \varphi(g_2) \in H$ ~~But~~

  so $\varphi(g_1) \cdot \varphi(g_2) = \varphi(g_1 g_2) \in H$
- If $g \in \varphi^{-1}(H)$, then $\varphi(g) \in H$, so $\varphi(g)^{-1} \in H$

  so $\varphi(g^{-1}) \in H$, so $g^{-1} \in \varphi^{-1}(H)$.

To every homomorphism $\varphi: G \to K$,

we can assign a special subgroup of G:

Def: For $\varphi: G \to K$ a homomorphism, define the kernel $\ker\varphi$ of φ to be

$$\ker\varphi = \{ g \in G \mid \varphi(g) = e_K \} = \varphi^{-1}(\{e_K\}).$$

By lemma above, we know $\ker\varphi$ is a subgroup.

Ex: (1) $\varphi: \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ as above.

$$\ker\varphi = \{ x \in \mathbb{Z} \mid \varphi(x) = \overline{x} = \overline{0} \}$$
$$= n\mathbb{Z}.$$

(2) $\ker \det = \{ A \in GL_n(\mathbb{R}) \mid \det A = 1 \}$