Lecture 20:

Theorem: No permutation in S_n can be expressed both as a product of an even number of transpositions, and as a product of an odd number of transpositions.

proof: We consider, for a permutation σ, the number of orbits of $(ij)\sigma$ vs the number of orbits of σ. And claim the difference is always 1.

case 1: i and j are in different cycles in σ (in the cycle decomposition of σ)

other cycles

$$\sigma = (a_1 \dots a_{k-1}\, i\, a_k \dots a_{n_1})(b_1 \dots b_{s-1}\, j\, b_{s+1} \dots b_{n_2})\,\sigma'$$

$$(ij)\sigma = (ij)(a_1 \dots a_{k-1}\, i\, a_k \dots a_{n_1})(b_1 \dots b_{s-1}\, j\, b_{s+1} \dots b_{n_2})\,\sigma'.$$

$$= (a_1 \dots a_{k-1}\, j\, b_{s+1} \dots b_{n_2}\, b_1 \dots b_{s-1}\, i\, a_k \dots a_{n_1})\,\sigma'.$$

So two cycles became 1.

case 2: i and j are in the same cycle in σ

$$\sigma = (a_1 \dots a_{k-1}\, i\, a_{k+1} \dots a_{s-1}\, j\, a_{s+1} \dots a_m)\,\sigma'$$

$$(ij)\sigma = (ij)(a_1 \dots a_{k-1}\, i\, a_{k+1} \dots a_{s-1}\, j\, a_{s+1} \dots a_m)\,\sigma'$$

$$= (a_1 \dots a_{k-1}\, j\, a_{s+1} \dots a_m)(i\, a_{k+1} \dots a_{s-1})\,\sigma'$$

one cycle became two! 1

conclusion: since the number of orbits changes by 1 each time we multiply with a new transposition, the parity of the number of transpositions changes the same number of times as the parity of the number of orbits. So you can't have the same permutation expressed in two ways as even or odd number of transpositions. $\square$

Def: A permutation is called <u>even</u> if it is the product of an even number of transpositions and <u>odd</u> if it is the product of an odd number of transpositions.

Def: The subset of even permutations in S_n is a subgroup called the alternating group (called A_n).

Q: What is the order of A_n?

(exercise: prove that $|A_n| = \frac{n!}{2}$ by showing that multiplying by (12) gives a bijection between the sets of even and odd permutations in S_n.)