Lecture 6:

Last time:
- Complex numbers
- Fundamental theorem of algebra (every $f(x) \in \mathbb{C}[x]$ complex poly of degree d has d roots (counted with multiplicity))

Fun fact: If $p(x) \in \mathbb{R}[x]$ is a polynomial with real coefficients, then the roots of $p(x)$ come in pairs $z, \bar{z}$.

proof: If $p(z) = a_0 + a_1 z + \dots + a_d z^d = 0$.

then $\overline{p(z)} = \bar{a}_0 + \bar{a}_1 \bar{z} + \dots + \bar{a}_d \bar{z}^d$.

$$= a_0 + a_1 \bar{z} + \dots + a_d \bar{z}^d = p(\bar{z}).$$

so $0 = \overline{p(z)} = p(\bar{z})$. so $\bar{z}$ is also a root!

Table of a binary operation

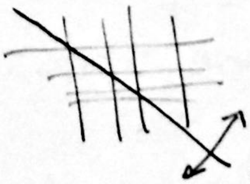
Example: Consider $\mathbb{Z}/5\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ with addition operation from before.

table:

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

Q: can you read if a binary operation is commutative from its table?

A: yes: the table will be symmetric



symmetry.

Isomorphic binary structures.

Let us call a pair $(S, *)$, where S is a set and $*$ is a binary operation a ~~binary algebraic~~ set-operation pair.

(not a standard term)

- Consider the pair: $\mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ with addition $(\mathbb{Z}/4\mathbb{Z}, +)$.
- consider also $U_4 = \{1, i, i^2 = -1, -i\}$ with multiplication $(U_4, *)$

Let's write the tables:

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

$*$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

these are kind of the same

they are identical if we re-label.

<u>Definition</u>: Let $(S_1, *_1)$, $(S_2, *_2)$ be set-operation pairs.

$\phi: S_1 \to S_2$ is an isomorphism if

ϕ is a bijection and "ϕ "respects" the operations".

$\forall x, y \in S_1$, $\quad \phi(x *_1 y) = \phi(x) *_2 \phi(y)$.

<u>Ex</u>: $(\mathbb{Z}/4\mathbb{Z}, +)$ and $(\{1, i, -1, -i\}, *)$

$$\phi: \mathbb{Z}/4\mathbb{Z} \longrightarrow \{1, i, -1, -i\}$$

$$\phi(\bar{0}) = 1$$
$$\phi(\bar{1}) = i$$
$$\phi(\bar{2}) = -1$$
$$\phi(\bar{3}) = -i$$

we can look at the table to check that

$$\phi(x+y) = \phi(x) * \phi(y)$$

or observe:

$$\phi(a) = e^{i a \cdot \pi/2}$$

so
$$\begin{aligned} \phi(a_1 + a_2) &= e^{i(a_1+a_2)\pi/2} \\ &= e^{i a_1 \pi/2} \cdot e^{i a_2 \pi/2} \\ &= \phi(a_1) \cdot \phi(a_2). \end{aligned}$$