$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

Math 120A Lecture 3:

Lemma: $f : A \to B$.

(1) f is injective iff f has a left-inverse.

(left inverse means a function $g : B \to A$ s.t. $g \circ f = id_A$)

(2) f is surjective iff f has a right-inverse.

(i.e. a function $g : B \to A$ s.t. $f \circ g = id_B$)

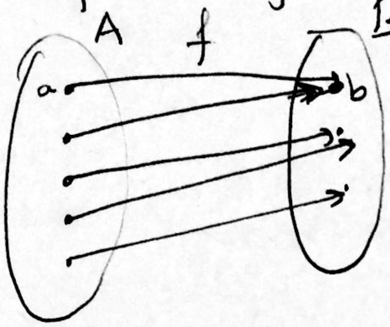
(3) f is bijective iff f has an inverse

(i.e. two-sided inverse, i.e $g : B \to A$ st $g \circ f = id_A$

$f \circ g = id_B$).

proof: (1) In homework.
(3) Exercise.

(2) Assume f is surjective



We want to show that there is a $g: B \to A$
s.t. $\forall b \in B \quad f \circ g(b) = f(g(b)) = b$.
Let, for each $b \in B$, $g(b)$ be ~~s.t.~~
one of the elements a in A s.t. $f(a) = b$.
(We are choosing lots of these elements so we are using the axiom of choice here)
Then we automatically have

$$f(\underbrace{g(b)}_{a}) = b.$$

Conversely, assume f has a right inverse g.
We know $\forall b \in B$, $f(g(b)) = b$.
Then f is surjective because, $\forall b \in B$,

$$f(\underbrace{g(b)}_{\text{this element maps to } b \text{ under } f}) = b$$

So f is surjective.

<u>Cardinality</u>: $|A|$ means # of elements in A.
number

But we can make sense of size for infinite sets by saying:

<u>Def</u>: A and B have the same cardinality if there exists a bijection $f: A \to B$.

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ have the same cardinality.
- $\mathbb{R}, [0,1], \mathbb{C}, \mathbb{R}^n$ have the same cardinality.
- $\mathbb{R}$ and $\mathbb{N}, \mathbb{Z}$, don't have the same cardinality.
- For any set S, S and $\mathcal{P}(S)$ don't have the same cardinality.

<u>Relations</u>: A relation between A and B is a subset of $A \times B$.

A relation on A is a subset of $A \times A$.

If $\sim$ is the relation $\sim \subset A \times A$, we often write $a \sim b$ for $(a,b) \in \sim$.

<u>Def</u>: A relation $\sim$ on A is an equivalence relation if

(1) $\sim$ is reflexive, i.e. $a \sim a$ $\forall a \in A$

(2) $\sim$ is symmetric i.e.

$\forall a,b, \quad a \sim b \Rightarrow b \sim a$

(3) $\sim$ is transitive, i.e. $\forall a,b,c$

$a \sim b$ and $b \sim c \Rightarrow a \sim c$.

Just like the properties of "=" on a set.

<u>Ex:</u> • On $\mathbb{Z}$, we have the relation

$a \sim b \Leftrightarrow n \mid b-a$. (for fixed n)

(homework problem to show this is an equivalence relation) eg $n=5$.

• On set of all triangles in $\mathbb{R}^2$, similarity is an equivalence relation.

<u>Set of equivalence classes</u>

$A/_\sim$ = "the set of equivalence classes of $\sim$".

eg: on $\mathbb{Z}$, $a \sim b \Leftrightarrow 5 \mid b-a$.

equivalence classes are
$\{0, 5, 10, \dots, -5, -10, \dots\} = \overline{0}$
$\{1, 6, 11, \dots, -4, -9, \dots\} = \overline{1}$
$\{2, 7, \dots, -3, -8, \dots\} = \overline{2}$
$\{3, \dots\} = \overline{3}$
$\{4, \dots\} = \overline{4}$

$\mathbb{Z}/_\sim$ is $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$