HOMEWORK 5

Due Thursday, May 18, at the beginning of discussion

1. An *automorphism* of a group G is a group isomorphism $\phi : G \to G$. List the automorphisms $\mathbb{Z} \to \mathbb{Z}$ and prove that these are the only automorphisms.
2. Remember to prove your answers in this problem:
   (a) Compute the lattice of subgroups of the group $\mathbb{Z}/45\mathbb{Z}$ and draw the lattice of subgroups.
   (b) Compute the lattice of subgroups of $\mathbb{Z}/2^n\mathbb{Z}$?
3. We know that every subgroup of $\mathbb{Z}$ is cyclic. Find the generators of the following subgroups and prove your answers.
   (a) $a\mathbb{Z} + b\mathbb{Z}$
   (b) $a\mathbb{Z} \cap b\mathbb{Z}$
4. Let $(G, \star)$ be a group. Fix an element a in G. The *centralizer* of a in G consists of all the elements of G that commute with a. Namely,

$$C(a) = \{g \in G \colon g \star a = a \star g\}.$$

   (the set of all elements x in G that commute with a).
   (a) Compute the centralizer of 3 in $(\mathbb{Z}, +)$.
   (b) Compute the centralizer of the element $r \in D_6$; and the centralizer of the element $r \in D_8$.
   (c) Show that for every group G and every element a in G, $C(a)$ is a subgroup of G.
   (d) In general, which one is bigger, $Z(G)$ and $C(a)$? Is one a subgroup of the other?
5. Consider the set $H = \{\beta \in S_5 \colon \beta(1) = 1 \text{ and } \beta(3) = 3\}$. (Recall that S_5 consists of bijective functions from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$.)
   (a) Show that H is a subgroup of S_5.
   (b) Show that H is isomorphic to S_3.
6. Write down a an isomorphism from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to a subgroup of S_4. You don't need to prove that the map you wrote is an isomorphism.
7. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$.
   (a) Compute α^{-1}, $\alpha\beta$
   (b) Compute α, β and $\alpha\beta$ as product of disjoint cycles
   (c) Compute α and β as product of 2-cycles.
8. Find the order of the following permutations:
   (a) $(145)(2345) \in S_5$.
   (b) $(154)(254)(1234) \in S_5$.
   (c) $(1574)(324)(3256) \in S_7$.

(d) $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 3$, as an element of $S_{\mathbb{Z}}$, the group of bijective maps $\mathbb{Z} \to \mathbb{Z}$ with composition as the operation.

9. What is the maximum order of an element in each of the groups S_4, S_5, S_6, S_7, S_8? Exhibit such an element in each case.