HOMEWORK 4

Due Thursday, May 4, at the beginning of discussion

1. Write the Cayley table for the Dihedral group D_8 with 8 elements. Why is D_8 not isomorphic to $\mathbb{Z}/8\mathbb{Z}$?
2. Describe the following sets:
   - $4\mathbb{Z} \cap 6\mathbb{Z} = \{x \in \mathbb{Z} \colon x \in 4\mathbb{Z} \text{ and } x \in 6\mathbb{Z}\}$
   - $4\mathbb{Z} \cup 6\mathbb{Z} = \{x \in \mathbb{Z} \colon x \in 4\mathbb{Z} \text{ or } x \in 6\mathbb{Z}\}$
   - $4\mathbb{Z} + 6\mathbb{Z} = \{h + k \colon h \in 4\mathbb{Z} \text{ and } k \in 6\mathbb{Z}\}$.

   For each of set S, either prove that the set is a subgroup of $(\mathbb{Z}, +)$, or disprove the claim by means of a counterexample.
3. Let $(G, \star)$ be a group and let H and K be subgroups of G. Prove or disprove each of the following statements. (If it is true, give a proof; if it is false, give a counterexample.)
   (a) $H \cap K = \{x \in G \colon x \in H \text{ and } x \in K\}$ is a subgroup of G.
   (b) $H \cup K = \{x \in G \colon x \in H \text{ or } x \in K\}$ is a subgroup of G.
   (c) (removed to be added to a later assignment)
   (d) Show that if $(G, \star)$ is abelian, then $H \star K$ is a subgroup of G.
   (e) If $G = \mathbb{Z}$, $H = 4\mathbb{Z}$ and $K = 6\mathbb{Z}$, find s, $t \in \mathbb{N}$ such that $H \cap K = s\mathbb{Z}$ and $H + K = t\mathbb{Z}$. How are s and t related to 4 an 6? In general, for $H = m\mathbb{Z}$ and $K = n\mathbb{Z}$, make a conjecture about $H \cap K$ and $H + K$.
4. (postponed to a later assignment)
   (a) Compute the list of subgroups of the group $\mathbb{Z}/45\mathbb{Z}$ and draw the lattice of subgroups. (prove that you really got all the subgroups)
   (b) How many subgroups does $\mathbb{Z}/2^n\mathbb{Z}$ have?
5. Let $(G, \star)$ be a group and let $G_2 = \{x \in G \colon x \star x = e\}$.
   (a) Find G_2 for $G = (\mathbb{Z}, +)$, $(\mathbb{R} - \{0\}, \cdot)$, $(\mathbb{Z}_6, +_6)$ and D_6. (You may refer back to their group table.)
   (b) Disprove the following theorem: For every group G, G_2 is a subgroup of G. (Make sure you explain why your counterexample disproves the theorem.)
   (c) Complete the theorem: If G is ________, G_2 is a subgroup of G. Then prove your theorem.
   (d) (optional) Suppose G Is abelian. Is $G_3 = \{x \in G \colon \ \colon x \star x \star x = e\}$ a subgroup as well? What about G_n (for $n \in \mathbb{N}$)?
6. Let $GL_2(\mathbb{R})$ be the group of 2×2 invertible matrices, with multiplication. (The elements of $GL_2(\mathbb{R})$ have real entries and non-zero determinant.) Consider the matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

a) Find the cyclic subgroup H of $GL_2(\mathbb{R})$ generated by the matrix A:

$$H = \langle A \rangle = \{A^k : k \in \mathbb{Z}\}.$$

b) Find a familiar group isomorphic to H. Explicitly provide an isomorphism (and check that the given map is, indeed, an isomorphism).

7. Let $(G, \star)$ and $(K, \circ)$ be groups. Let $\phi\colon G \to K$ be a group homomorphism (not necessarily an isomorphism). Prove that

(a) $\phi(e_G) = e_K$, so ϕ maps the identity of G into the identity of K.

(b) Show that for all $g \in G$, $\phi(g^{-1}) = [\phi(g)]^{-1}$.
That is, ϕ maps the inverse of g into the inverse of $\phi(g)$.