HOMEWORK 3

Due Thursday, April 27, at the beginning of discussion

1. (moved to next homework) Write the Cayley table for the Dihedral group D_8 with 8 elements. Why is D_8 not isomorphic to $\mathbb{Z}/8\mathbb{Z}$?
2. Prove or disprove: If G is a group and $g, h \in G$, then $(gh)^{-1} = h^{-1}g^{-1}$.
3. Prove or disprove: If G is a group, $(g^{-1})^{-1} = g$.
4. Which of the following set-operation pairs are groups? All matrices are with real entries. Please prove each answer but feel free to cite any facts from linear algebra.
   (a) $n \times n$ matrices with matrix addition.
   (b) $n \times n$ matrices with matrix multiplication.
   (c) $n \times n$ diagonal matrices with matrix multiplication.
   (d) $n \times n$ diagonal matrices with no zero-entries in the diagonal, with matrix multiplication.
   (e) $n \times n$ matrices with non-zero determinant (this is called $\mathrm{GL}_n(\mathbb{R})$), with matrix multiplication.
   (f) $n \times n$ matrices with determinant $+1$, with matrix multiplication.
   (g) $n \times n$ orthogonal matrices (an orthogonal matrix is a matrix whose columns are ortho-normal to each other, equivalently $A^T A = I$), with matrix multiplication.
5. Prove that the intersection of two subgroups is always a subgroup. i.e. If G is a group and H and K are subgroups of G, then $H \cap K$ is also a subgroup of G.
6. Give an example of two groups with 9 elements each which are not isomorphic to each other (and prove it).
7. Assume that G is a group such that for all $x \in G$, $x * x = e$. Prove that G is an abelian group.
8. Let $(G, \star)$ be a group. Define the **center** of G by

$$Z(G) := \{x \in G \colon x \star a = a \star x,\ \forall a \in G\}.$$

   The set $Z(G)$ consists of all elements of G that commute with every possible element of the group. For example, one can say that the matrix $4I$ belongs to the center of $(GL(2, \mathbb{R}), \cdot)$ because $(4I)A = A(4I)$ for all A in $GL(2, \mathbb{R})$, since both sides are equal to $4A$.
   (a) Show that, for every group G, the center $Z(G)$ is a subgroup of G.
   (b) Find the center of $(\mathbb{Z}_4, +)$ and (this part is moved to next homework) the center of D_6, the dihedral group. (You should be able to tell from the group table.)
   (c) One could say that "the center $Z(G)$ measures the abelian-ness of a group G". Please interpret this statement.[1]

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[1]Recall that a group $(G, \star)$ is called abelian if the operation $\star$ is commutative. *Hint*: What is $Z(G)$ equal to when G is abelian?