MATH 3A: HOMEWORK 4

Due Tuesday, Nov 1st, at the beginning of your discussion session

1. Find the matrix of ?s.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$$

2. Find the standard matrix for each of the following transformations. Find out if each of them is one-to-one, onto. (or both, or neither)

$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$

$T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

3. Suppose vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \dots, p$. Show that T is the zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

4. Which of the following matrices can be multiplied? (be careful, the order in which you are trying to multiply them changes the answer!) Multiply a few of them abut indicate for every two whether they can be multiplied.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

5. Find the standard matrix for the following transformations.

$T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$, and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(\mathbf{e}_1) = (1, 4)$, $T(\mathbf{e}_2) = (-2, 9)$, and $T(\mathbf{e}_3) = (3, -8)$, where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

6. True or false?
   - The range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A.
   - A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?
   - A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ always maps the origin of $\mathbb{R}^n$ to the origin of $\mathbb{R}^m$.

7. Let $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

8. Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

9. Find the inverses of the following matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

10. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.

11. If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second columns of B.

12. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

13. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

14. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

15. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

16. True or False: If an $n \times n$ matrix A has zeroes everywhere except for a single 1 in each row and column, then A is invertible.