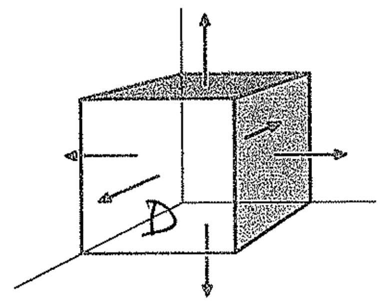
MATH 240 PROBLEMS

Solutions

1

10. Let S be the surface consisting of the boundary of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and let $\mathbf{F} = (e^x + z)\mathbf{i} + (y^2 - x)\mathbf{j} - xe^y\mathbf{k}$. Evaluate the outward flux (or divergence)

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \oiint_S \mathbf{F} \cdot \mathbf{n}\, dS\,,$$



also written

$$\oiint_S (e^x + z)\, dy\, dz + (y^2 - x)\, dz\, dx - xe^y\, dx\, dy\,.$$

(A) $e - 2$ (B) e (C) $e + 2$ (D) $e + 3$ (E) $4 - e$

We use the divergence theorem!.

$$\iint_S F \cdot \vec{n}\, dS = \iiint_D \operatorname{div} F\, dV \quad \text{where } D \text{ is the cube}$$

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = e^x + 2y + 0$$

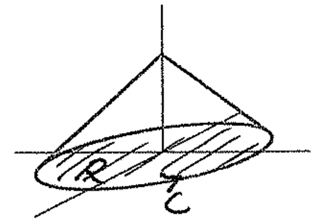
$$\iiint_D \operatorname{div} F = \int_0^1 \int_0^1 \int_0^1 (e^x + 2y)\, dx\, dy\, dz = \int_0^1 \int_0^1 (e^x + 2xy)\Big|_0^1 dy\, dz$$

$$= \int_0^1 \int_0^1 (e + 2y - 1)\, dy\, dz = \int_0^1 (e - 1 + 1)\, dz = e$$

2

20. Let $\mathbf{F} = (x-y)\mathbf{i} - 2xz\mathbf{j} - x^2\mathbf{k}$. Evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$



where S is the portion of the cone $z = 1 - \sqrt{x^2+y^2}$ above the xy-plane with outward directed normal (away from the origin).

Equivalently, evaluate $\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$ where $\nabla \times \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

(A) 0 (B) π (C) 2π (D) 3π (E) 4π

By stokes' theorem, we know that $\iint_S \text{Curl}\,F \cdot \vec{n} = \oint_C F \cdot dr$

Again by Stokes' theorem, $\oint_C F \cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$\frac{\partial Q}{\partial x} = -2z \quad \frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2z + 1$

$\iint_R (-2z+1)\,dA = \iint_R 1\,dA = \pi$

$z = 0$ here. on R

(3)

11. Let Q be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$ and the xy-plane. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ where S is the surface of Q oriented by the outside normal, and $\mathbf{F}(x, y, z) = (x^2 + \sin z)\mathbf{i} + (xy + \cos z)\mathbf{j} + e^y\mathbf{k}$.

S includes the caps

The solid looks like this.

~~Apply~~ the divergence theorem:

$$\iint_S F \cdot \vec{n} = \iiint_D \operatorname{div} F \, dV$$

$$\operatorname{div} F = 2x + x + 0 = 3x$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{6-\cancel{x}} 3x\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{2}\int_0^{6-r\cos\theta} 3r^2\cos\theta \, dz\, dr\, d\theta$$

then evaluate, the answer should be -12π.

4

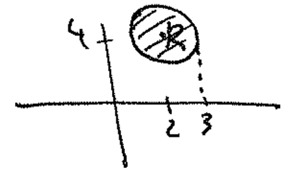
12. Evaluate

$$\oint_C \overbrace{(e^x + 3y)}^{P}\,dx + \overbrace{(4x + y^6)}^{Q}\,dy$$

where C is the circle $(x-2)^2 + (y-4)^2 = 1$, traversed counterclockwise.

(A) 0 (B) 1 (C) e (D) π (E) 4

Apply Green's theorem.



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4 - 3 = 1$$

So $\displaystyle\oint_C P\,dx + Q\,dy = \iint_R 1\,dA = \pi$

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8. Find c_4, where $y = \sum_{k=0}^{\infty} c_k x^k$ is a solution to the differential equation

$$y'' - (1+x^2)y = 0\,,$$

subject to the initial condition $y(0) = 8$ and $y'(0) = -3$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

$y = \sum_{k=0}^{\infty} c_k x^k \qquad y' = \sum_{k=1}^{\infty} k c_k x^{k-1} \qquad y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$

plug in to the equation :

$$\sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} c_k x^{k+2} = 0$$

$\{ n = k-2 \qquad \{ n = k+2$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$$(2.1.c_2 - c_0)x^0 + (3.2.c_3 - c_1)x + \sum_{n=2}^{\infty} \Big[\underbrace{(n+2)(n+1)c_{n+2} - c_n - c_{n-2}}_{=\,0 \text{ is the recurrence relation}}\Big] x^n = 0$$

all terms in this power series must be 0,

so $\quad 2c_2 - c_0 = 0$

$\quad 6c_3 - c_1 = 0$

$(n+2)(n+1)\, c_{n+2} = c_n + c_{n-2}$

$c_{n+2} = \dfrac{c_n + c_{n-2}}{(n+2)(n+1)}$

then $c_2 = 4$

and $c_4 = \dfrac{c_2 + c_0}{4 \cdot 3} = 1$

$y = c_0 + c_1 x + c_2 x^2 + \ldots \Rightarrow y(0) = c_0 = 8$

$y' = c_1 + 2c_2 x + \ldots \Rightarrow y'(0) = c_1 = -3$

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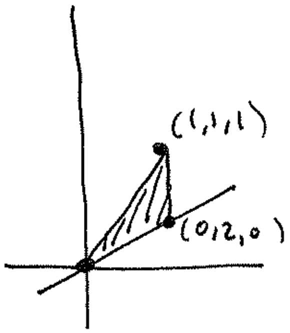
10. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = 2y\mathbf{i} + 3z\mathbf{j} + x\mathbf{k}$$

and C is the triangle with vertices $(0,0,0)$, $(0,2,0)$ and $(1,1,1)$ oriented so that the vertices are traversed in that order.



We want to use Stokes' theorem. $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{Curl}\,\mathbf{F}\cdot\vec{n}$

We need: Curl F and $\vec{n}$

$$\text{Curl}\,\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (-3, -1, 0)$$

$\vec{n}$ is normal to both the vectors $(0,2,0)$ and $(1,1,1)$

take cross product $\begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = (2, 0, -2)$

(check that this has correct orientation)

So $\vec{n} = \frac{1}{2\sqrt{2}}(2, 0, -2)$ after normalizing.

So $\iint \text{curl}\,\mathbf{F}\cdot\vec{n} = \iint \frac{-6}{2\sqrt{2}} = \frac{-3}{\sqrt{2}}\iint 1\,dA = -3$

the area of the triangle is equal to $\sqrt{2}$, you can see this from the fact that the area of the parallelogram whose vertices are 0, v, w, w+v is v×w

So $|(0,2,0)\times(1,1,1)|$ gives twice the area of the triangle.

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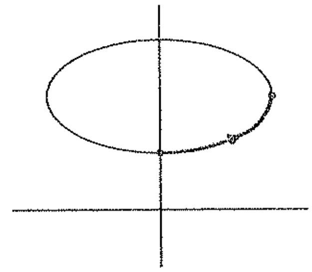
11. Find

$$\int_C \underbrace{(2x + e^y)}_{P}\, dx + \underbrace{(3y^2 + xe^y)}_{Q}\, dy\,,$$

where C is the arc of the ellipse $x^2 + 4(y-2)^2 = 4$ from $(0,1)$ to $(2,2)$ in the counterclockwise direction.



(A) $e^3 + 12$ (B) $3e - 7$ (C) $2e^2 + 11$ (D) $3e^2 + 14$ (E) $e^2 + 8$

$$\frac{\partial Q}{\partial x} = e^y = \frac{\partial P}{\partial y}$$ (P, Q) is conservative.

need to find ϕ s.t $(P, Q) = \nabla \phi$

$$\phi = x^2 + xe^y + y^3$$

plug in the endpoints

$$\int_C P\,dx + Q\,dy = \int_C \nabla\phi \cdot dr = \phi(2,2) - \phi(0,1)$$

$$= 4 + 2e^2 + 8 - 1 = 2e^2 + 11$$

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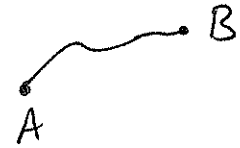
9. Evaluate

$$\int_C \overbrace{(1-ye^{-x})}^{P}dx + \overbrace{e^{-x}}^{Q}dy$$



where C is the curve $r(t) = (e^{cos\pi t}, 1/(t^7+1))$, $0 \le t \le 1$.

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ So the integral only depends on the endpoints

$\phi = x + ye^{-x}$ then $D\phi = (P, Q)$

$A = r(0) = (e, 1)$

$B = r(1) = (e^{-1}, \frac{1}{2})$

$\int_C D\phi\, dr = \phi(B) - \phi(A)$

$= e^{-1} + \frac{1}{2}e^{-e^{-1}} - e - e^{-e}$? Whaaaat!?

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8. While subject to force $\mathbf{F}(x, y) = y^3\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$, a particle travels once around the circle of radius 3. Use Green's Theorem to find the work done by $\mathbf{F}$.

$$\int_C F \cdot dr = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\frac{\partial Q}{\partial x} = 3x^2 + 3y^2 \qquad \frac{\partial P}{\partial y} = 3y^2$$

$$\text{so } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2$$

$$\int_C F \cdot dr = \iint_{\substack{\text{disk} \\ \text{of radius } 3}} 3x^2 \, dA = 3 \cdot \int_0^{2\pi} \int_0^3 3r^2 \cos^2\theta \; r \, dr \, d\theta \quad \ldots$$

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17. The value of the surface integral $\iint_S F \cdot n\, dA$, where $F = \frac{x^3}{3} i + \frac{y^3}{3} j + z^2 k$, and S is the closed cylindrical shell $x^2 + y^2 = 4, 0 \le z \le 3$ (including the top and bottom disks), oriented by the outwards normal is:

a) 0 *b*) 60π *c*) -60π *d*) 12π *e*) -12π *f*) 10 *g*) 36π

Use the Divergence theorem:

$$\iint_S F \cdot \vec{n}\, dA = \iiint_{\text{cylinder}} \operatorname{div} F$$

$$\operatorname{div} F = x^2 + y^2 + 2z$$

$$= \int_0^{2\pi} \int_0^2 \int_0^3 (r^2 + 2z)\, r\, dz\, dr\, d\theta$$

(11)

they also implicitly assume that F is defined everywhere.

5. Suppose that $F = M(x,y)\mathrm{i} + N(x,y)\mathrm{j}$, M and N have continuous partial derivatives and C is a smooth closed curve enclosing a region D.
Indicate whether each expression is defined and for those which are defined, label each statement as true or false:

a) If $\iint_D (N_x - M_y) = 0$, then $\int_C \mathbf{F} \cdot d\mathrm{r} = 0$ true

b) div ($\mathbf{F}$) is a vector. false, it's a function.

c) If $\mathbf{F} = \nabla f$ and E is a curve starting at P_1 and ending at P_2, then: $\int_E \mathbf{F} \cdot \mathrm{r} = \mathbf{F}(P_1) - \mathbf{F}(P_2)$

the order is wrong FALSE

d) $curl(\underbrace{div(\mathbf{F})}_{\text{function}}) = 0$. not defined

curl is defined for a vector field only.

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6. The direction of the steepest ascent at $P = (3, 0)$ of the mountain $f(x, y) = 4 - \frac{2}{3}\sqrt{x^2 + y^2}$ is:

$a)$ $\mathrm{i}+\mathrm{j}$ $\quad b)$ $-\frac{2}{3}\mathrm{i}$ $\quad c)$ $-\frac{2}{3}\mathrm{i}+\mathrm{j}$ $\quad d)$ 0 $\quad e)$ $-\frac{2}{3}\mathrm{j}$ $\quad f)$ $-\frac{2}{3}\mathrm{j}$

this is given by the Gradient.

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j = \left(-\frac{2}{3}\,\frac{\frac{1}{2}\,2x}{\sqrt{x^2+y^2}}, -\frac{2}{3}\,\frac{\frac{1}{2}\,2y}{\sqrt{x^2+y^2}}\right)$$

$$\nabla f(3,0) = \left(-\frac{2}{3}, 0\right)$$

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9. Find a recurrence relation for c_n, where $y = \sum_{k=0}^{\infty} c_k x^k$ is a solution to the differential equation

$$y'' + xy' + y = 0\,.$$

That is, give a formula for c_{n+2} in terms of c_n and/or c_{n+1}.

(A) $c_{n+2} = -\dfrac{c_n + c_{n+1}}{n}$ (B) $c_{n+2} = \dfrac{(2n-1)}{(n+1)(n+2)} c_n$ (C) $c_{n+2} = -\dfrac{c_n}{n+2}$

(D) $c_{n+2} = -\dfrac{c_n + c_{n+1}}{n(n+1)}$ (E) $c_{n+2} = -\dfrac{c_n}{2n(n-1)}$

$$y = \sum_{k=0}^{\infty} c_k x^k \qquad y' = \sum_{k=1}^{\infty} k c_k x^{k-1} \qquad y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$$

$$y''+xy'+y = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$n = k-2$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \text{same} + \text{same}$$

we get $(n+2)(n+1) c_{n+2} + n c_n + c_n = 0$

$$c_{n+2} = \frac{-c_n}{n+2}$$

(14)

15. Let P_1 and P_2 be two points in three-space, and C a curve joining P_1 to P_2. For what values of a is the line integral $\int_C 3x^2y^5dx+ax^3y^4dy+dz$ independent of C ?

a) 5 *b)* -5 *c)* 3 *d)* -3 *e)* 1 *f)* 0 *g)* *novalues*

For this, we need Curl F = 0.

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Notice that R depends only on z and P and Q only on x and y.

This makes the i and j components vanish.

The k component of curl F is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3ax^2y^4 - 3x^2 \cdot 5y^4 = 0$$

$$\Rightarrow \quad a = 5$$

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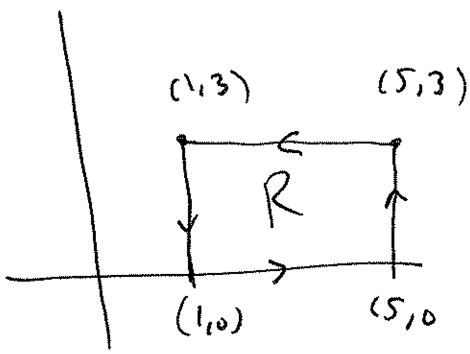
16. The value of the line integral $\int_C (-y + cos(x^2))dx + (3x + e^{\sqrt{y^2-1}})dy$ where C is the boundary of the rectangle with vertices at $(1,0), (1,3), (5,0), (5,3)$ oriented counterclockwise is:

$a)$ 12 $\quad$ $b)$ 15 $\quad$ $c)$ 24 $\quad$ $d)$ 12π $\quad$ $e)$ 48 $\quad$ $f)$ -6π $\quad$ $g)$ 0



Use Green's theorem:

$$\int_C F \cdot dr = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$= \iint_R (3 - (-1))\, dA$$

$$= 4 \cdot \underbrace{\iint_R 1\, dA}_{\text{area of } R = 12} = 48$$

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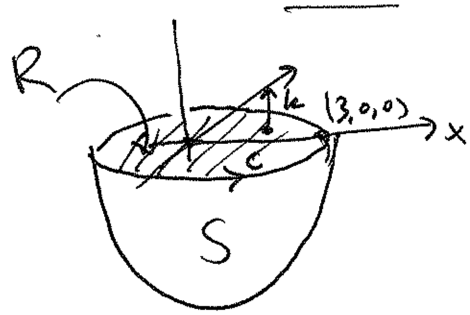
18. Let $F = \mathbf{i} + 3x\mathbf{j} + e^{sin(2x)}\mathbf{k}$ be a vector field. The value of $\int\int_S (\nabla \times F) \cdot n dA$ over the surface $z = x^2 + y^2 - 9, z \leq 0$, oriented by the normal pointing "upward" (i.e. in the positive z direction) is:

$a)$ 3π $\quad$ $b)$ 27π $\quad$ $c)$ 6π $\quad$ $d)$ 0π $\quad$ $e)$ -9π $\quad$ $f)$ 9π $\quad$ $g)$ $e^{2\pi}$



Stokes' theorem gives

$$\iint_S \text{curl}F \cdot \vec{n} = \oint_{\substack{\text{circle}\\ \text{of radius } 3}} F \cdot dr$$

Use Stokes' theorem again:

$$\oint_C F \cdot dr = \iint_R \text{curl}F \cdot \vec{n} = \iint_{\text{disc of } R=3} 3\, dA = 3.\pi 3^2 = 27\pi$$

need only the k component

$\vec{n}$ = (0,0,1)

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8. Let S be the closed surface in 3-space formed by the cone

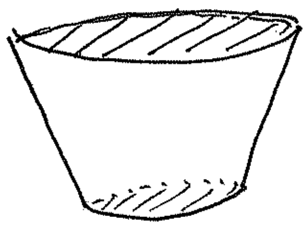
$$x^2 + y^2 - z^2 = 0, \quad 1 \le z \le 2,$$

the disk $x^2 + y^2 \le 4$ in the plane $z = 2$, and the disk $x^2 + y^2 \le 1$ in the plane $z = 1$. Define the vector field

$$\mathbf{F}(x, y, z) = xy^2\mathbf{i} + x^2y\mathbf{j} + \sin x\,\mathbf{k}$$

and let $\mathbf{n}$ be the outward pointing unit normal vector to S. Compute the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$



the surface is already closed.

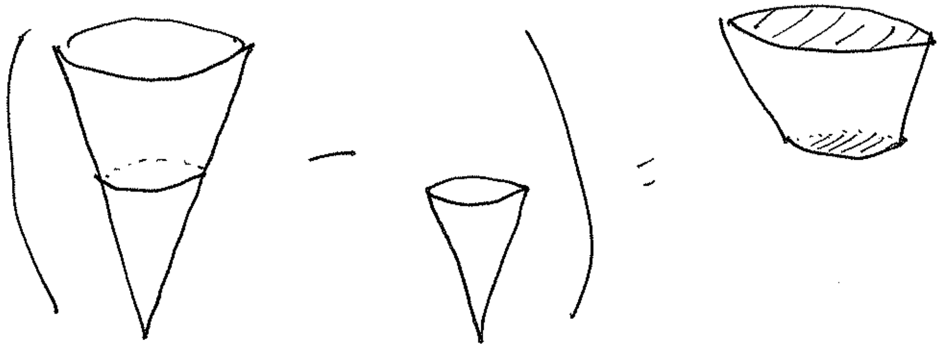
By the divergence theorem.

$$\iint_S F \cdot \vec{n}\, dS = \iiint \operatorname{div} F\, dv$$

$x^2+y^2 = r^2$ in polar coords.

$$\operatorname{div} F = y^2 + x^2 + 0$$

write in polar coordinates and do it in two pieces.



$$\int_0^{2\pi}\int_0^2\int_r^2 r^2 \underbrace{r\,dz\,dr\,d\theta}_{dV} - \int_0^{2\pi}\int_0^1\int_r^1 r^2\, r\,dz\,dr\,d\theta$$ then evaluate.

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7. Define the function

$$f(x,y,z) = e^{(\sin x \cdot \cos y)} \cdot \left(z + \frac{\pi}{2}\right).$$

Let C be the curve

$$r(t) = (t\cos^2(2t),\ t\sin(t),\ t)$$

for $0 \le t \le \pi/2$. Compute the integral

$$\int_C \frac{\partial f}{\partial x}\,dx + \int_C \frac{\partial f}{\partial y}\,dy + \int_C \frac{\partial f}{\partial z}\,dz.$$

$$\int_C \nabla f \cdot dr = f(\underset{\uparrow}{B}) - f(\underset{\nearrow}{A})$$

endpoints

$$A = r(0) = (0,0,0)$$

$$B = r(\pi/2) = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$f(B) = e^0 \cdot \pi$$

$$f(A) = \cancel{1}\, e^0\left(\frac{\pi}{2}\right)$$

$$f(B) - f(A) = \frac{\pi}{2}$$

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5. Consider the vector field

$$\mathbf{F} = \left(\frac{-z^2}{5} - z + \pi y e^{\sin x} \cos x \right) \mathbf{i} + \left(\pi e^{\sin x} - x \right) \mathbf{j} - \frac{2xz}{5} \mathbf{k}$$

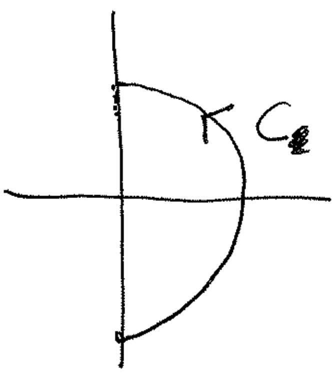
and the curve C given by

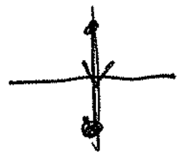
$$(2\cos t, 2\sin t, 0)$$

for $-\pi/2 \leq t \leq \pi/2$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(a) $2\pi\sqrt{2}$ (b) 0 (c) 4π (d) $-\pi$ (e) -2π (f) 2π (g) none of the above



add to this:  to make it a closed curve, then use Green's theorem.

Official solution on the next page.

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5. Consider the vector field

$$\mathbf{F} = \left(\frac{-z^2}{5} - z + \pi y e^{\sin x}\cos x\right)\mathbf{i} + \left(\pi e^{\sin x} - x\right)\mathbf{j} - \frac{2xz}{5}\mathbf{k}$$

and the curve C given by

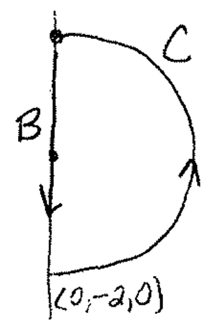
$$(2\cos t, 2\sin t, 0)$$

for $-\pi/2 \le t \le \pi/2$. Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

(a) $2\pi\sqrt{2}$ (b) 0 (c) 4π (d) $-\pi$ (e) -2π (f) 2π (g) none of the above

$$\nabla\times F = \hat{i}\left(0 - 0\right) - \hat{j}\left(-\frac{2z}{5} + \frac{2z}{5} + 1\right) + \hat{k}\left(-1\right) = -\hat{j} - \hat{k}.$$



$\vec{n} = \hat{k}.$

$(\nabla\times F)\cdot\hat{k} = -1.$

$B(t) = \cancel{(0,0,-t)}\ (0,-t,0)$ for $-2 \le t \le 2.$

$$\int_C F\cdot dr + \int_B F\cdot dr = \iint_{\substack{\text{half}\\ \text{disk}}} (-1)\cdot dA = \frac{\pi\cdot 2^2}{2}\cdot(-1) = -2\pi.$$

$$\int_B F\cdot dr = \int (\pi y\,\hat{i} + \pi\,\hat{j})\cdot\left(\frac{dy}{dt}\,dt\right)\hat{j} = -\pi\int_{-2}^{2} dt = -4\pi.$$

$$\Rightarrow \int_C F\cdot dr = 2\pi.$$

[redacted]

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8. The value of the line integral $\int_C \overbrace{xy^2}^{P}dx+\overbrace{x^2y}^{Q}dy$ over the curve C parametrized by $\underbrace{(1+cos^3(t))\mathbf{i}+(1-sin^6(t))\mathbf{j}}_{r(t)}$, $0 \le t \le \pi$, and oriented in the direction of increasing t is:

$a)$ $\pi/2$ $b)$ $-\pi$ $c)$ 2 $\underline{d)\ -2}$ $e)$ 0 $f)$ -1 $g)$ $-\pi^2/4$

$(P, Q) = \nabla\phi$ where $\phi = \frac{1}{2}x^2y^2$

$A = r(0) = (2, 1)$

$B = r(\pi) = (0, 1)$

$\int_C F.dr = \phi(B) - \phi(A)$

$= 0 - 2 = -2$