7 June

<u>Lecture 8</u>: • midterm: one week from now! cheat sheet allowed. 2 hours. extra office hours before. (on sat or sun?)

• no office hours today, double tomorrow.

Last time: • quiz (over 20/30 is good score) discuss

• you can turn in homework to be graded if you are not sure about things

• we started talking about ODEs, homogeneous equations, degree of an equation. general solution to homogeneous and non-homogeneous equations.

an ODE looks like

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = g(x)$$

The general solution looks like

$$y = \underset{\text{one particular solution}}{y_p} + \underbrace{c_1 y_1 + c_2 y_2 + \dots + c_n y_n}_{n \text{ solutions to the homogeneous version: } a_n(x) y^{(n)} + \dots + a_1(x) y' + a_0(x) = \underline{\underline{0}}}$$

(n = degree of the equation)

Linear homogeneous equations with constant coefficients.

These are equations that look like:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$a_0, \dots, a_n$ are just numbers.

eg: Solve $\quad y'' - 7y' + 12y = 0$

try: $\quad y = e^{mx} \qquad y' = me^{mx} \quad y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 7me^{mx} + 12e^{mx} = 0$$

$$e^{mx}(\underbrace{m^2 - 7m + 12) = 0}$$

this is called the auxiliary equation.

$$e^{mx}(m-3)(m-4) = 0$$

so we must have $m = 3$ or $m = 4$.

so the general solution is $y = c_1 e^{3x} + c_2 e^{4x}$.

This is the strategy in general, plug in e^{mx} and find m.

- What happens when there are double roots?

eg: $\quad y'' - 4y' + 4y = 0$

$y = e^{mx} \qquad e^{mx}(m^2 - 4m + 4) = e^{mx}(m-2)^2 = 0$

we still have our solution $y = e^{2x}$ but there must be another one. Try $y = xe^{mx}$.

$y' = e^{mx} + mxe^{mx} \qquad y'' = me^{mx} + me^{mx} + m^2 xe^{mx}$

$y'' - 4y' + 4y = m^2 xe^{mx} + 2me^{mx} - 4e^{mx} - 4mxe^{mx} + 4xe^{mx}$

$$= \underbrace{(m^2 - 4m + 4)}_{=0} x e^{mx} + \underbrace{2(m-2)}_{=0} e^{mx} = 0$$ it works.

(we'll see how we came up with xe^{mx})

So the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

What if we find complex roots?

eg: $y'' + 2y' + 10y = 0$

auxiliary equation: $m^2 + 2m + 10$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3\underset{\displaystyle i}{\sqrt{-1}}$$

$$= -1 \pm 3i$$

then the solution is:

$$y = e^{-x} (c_1 \cos(3x) + c_2 \sin(3x))$$

Summary: For a linear equation with constant coefficients, $ay'' + by' + cy = 0$, look at the roots of the auxiliary equation: $am^2 + bm + c = 0$

- distinct roots m_1, m_2: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
- repeated root: m_1, m_1: $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$
- complex roots $m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$ then

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

- How did we come up with $xe^{m_1 x}$ for the second solution in the repeated root case?

We used a <u>trick</u> called reduction of order.

The trick is as follows: if we already know one solution y_1 to a degree 2 equation, we can try to find another by plugging in $y_2 = u(x) \cdot y_1(x)$ $\quad y_2 = u \cdot y_1$

<u>example</u>: $\quad y'' - 4y' + 4y = 0$

auxiliary equation $\quad m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0 \quad \text{repeated root.}$$

we know that $y_1 = e^{2x}$ is one solution. Let's find the other by trying $y_2 = u \cdot e^{2x}$

plug in: $(ue^{2x})'' - 4(ue^{2x})' + 4ue^{2x} = 0$

$$(ue^{2x})' = u'e^{2x} + 2ue^{2x}$$

$$(ue^{2x})'' = u''e^{2x} + 2u'e^{2x} + 2u'e^{2x} + 4ue^{2x}$$

$$= u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$$

$$u''e^{2x} + \cancel{4u'e^{2x}} + 4ue^{2x} - \cancel{4u'e^{2x}} - 8ue^{2x} + 4ue^{2x} = 0$$

$$u''e^{2x} + 4u\big(\underbrace{e^{2x} - 2e^{2x} + e^{2x}}_{0}\big) = 0$$

$$\Rightarrow \quad u'' = 0 \qquad \text{so} \qquad u = c_2 x + c_3$$

so $\quad y_2 = e^{2x}(c_2 x + c_3) = \boxed{c_2 x e^{2x}} + c_3 e^{2x}$ this is already contained in y_1.

In general, when we plug in $y_2 = uy_1$, the fact that y_1 is already a solution gives us a similar cancellation. This gives us an equation in degree 2 that has u'' and u' in it but not u, so it's a degree 1 equation for $w = u'$. Then you can integrate to find u, once you have $w = u'$.

eg: $y'' + P(x)y' + Q(x)y = 0$

say y_1 is known to be a solution

plug in $y_2 = uy_1$

$$y_2' = u'y_1 + uy_1'$$
$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)u'y_1 + P(x)uy_1' + Q(x)uy_1 = 0$$

$$u''y_1 + 2u'y_1' + P(x)u'y_1 + u(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{0}) = 0$$

we're left with an equation only in u'

make a change of variables $w = u'$

$$w'y_1 + 2wy_1' + P(x)wy_1 = 0$$

degree 1 equation. easier to solve.

How to solve non-homogeneous linear equations.

eg: $\frac{1}{4}y'' + y' + y = x^2 - 2x$

- first, let's solve the homogeneous version

$$\frac{1}{4}y'' + y' + y = 0 \qquad \text{solution is of type } e^{mx}$$

$$y'' + 4y' + 4y = 0 \qquad \text{auxiliary:} \quad m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0$$

so solutions are $y_c = c_1 e^{-2x} + c_2 x e^{-2x}$

- now let's solve the nonhomogeneous equation:

guess! that the solution is $y = Ax^2 + Bx + C$

$$y' = 2Ax + B \qquad y'' = 2A$$

plug in: $\frac{1}{4}2A + 2Ax + B + Ax^2 + Bx + C = x^2 - 2x$

this means: $A = 1 \qquad 2A + B = -2$

so $B = -3$

and $\frac{1}{4}2A + B + C = 0$ so: $C = -\frac{5}{2}$

so $y_p = x^2 - 3x - \frac{5}{2}$

so $\boxed{y = x^2 - 3x - \frac{5}{2} + c_1 e^{-2x} + c_2 x e^{-2x}}$

eg: $y'' + 4y = \cos x$

· homogeneous equation: $y'' + 4y = 0$

auxiliary eq.: $m^2 + 4 = 0$

$m = 0 \pm 2i$

solutions $y_c = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$= c_1 \cos 2x + c_2 \sin 2x$

· $y_p = ?$ guess: $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$y'' + 4y = \underbrace{-A\cos x} - B\sin x + \underbrace{4A\cos x} + 4B\sin x = \cos x$

$3A\cos x + 3B\sin x = \cos x$

$A = \frac{1}{3} \quad B = 0$

$$\boxed{y = \frac{1}{3}\cos x + c_1 \cos 2x + c_2 \sin 2x}$$

• table of what to guess at the end of lecture notes. we'll fill it here together.

Funny situation:

eg: $y'' - 5y' + 4y = 8e^x$

to find y_c (solution to $y'' - 5y' + 4y = 0$)

auxiliary $m^2 - 5m + 4 = 0 \quad (m-4)(m-1) = 0$

so $y_c = c_1 e^{4x} + c_2 e^x$

ok, now for y_p. guess: $y = Ae^x$

plug in $\quad Ae^x - 5Ae^x + 4Ae^x = e^x \underbrace{(A - 5A + 4A)}_{0} = 8e^x$

uh-oh!

this cannot work because Ae^x is already a solution to the homogeneous version of the eqn.

try: $y = Axe^x$

$y' = Ae^x + Axe^x$

$y'' = 2Ae^x + Axe^x$

$y'' - 5y' + 4y = 2Ae^x + \underline{Axe^x} - 5Ae^x \underline{- 5Axe^x} + \underline{4Axe^x} = 8e^x$

$= \underbrace{Axe^x - 5Axe^x + 4Axe^x}_{0} + 2Ae^x - 5Ae^x = 8e^x$

$-3Ae^x = 8e^x$

$A = \frac{-8}{3}$

<u>double trouble</u>:

$y'' - 2y' + y = e^x$

$y_c = c_1 e^x + c_2 x e^x$

can't try xe^x because it is also a solution.

$x^2 e^x$ works! why?

$g(x)$	guess for y_p
1 (any constant)	A
$3x+1$	$Ax+B$
$5x^2+2x+3$	Ax^2+Bx+C
x^3-2x^2+x-5	Ax^3+Bx^2+Cx+D
$\vdots$	
$\sin 5x$	$A\cos 5x + B\sin 5x$
e^{5x}	Ae^{5x}
$(9x-1)e^{5x}$	$(Ax+B)e^{5x}$

in general, take product.

For solving second order equations, there is a "killer method" that can solve pretty much anything. It's called "variation of parameters". We won't do it, but know that it is there.