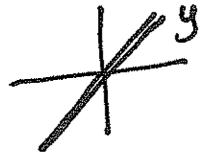
<u>Lecture 2</u>: 25 May

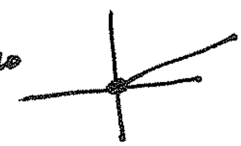
Previously on math 240:
- talked about how matrices multiply w/ vectors.
- defined a vector space. e.g. $\mathbb{R}^n$
- defined what a subspace is, ~~som~~ it was a subset of the vector space in consideration closed under addition and scalar multiplication.

remark: dot product not part of the data that's extra

~~eg~~: in $\mathbb{R}^2$



$y=x$ line is a subspace



no

no.

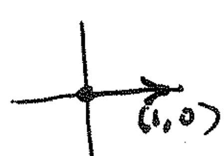
what about in $\mathbb{R}^3$. check:
- the xy plane
- the $y=0$, $z=0$ line
- the plane $ax+by+cz=0$.

but not a plane not going through the ~~o~~ origin.

<u>Span</u>: the span of vectors $\vec{v}_1, \dots, \vec{v}_k$ is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_k$. (subspace spanned by $\vec{v}_1, \dots, \vec{v}_k$)

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \{\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_k \vec{v}_k \mid \lambda_i \in \mathbb{R} \text{ for all } i = 1, \dots, k\}$$

<u>examples</u>: $\operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \subset \mathbb{R}^2$



is the x axis

$\operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \subset \mathbb{R}^2$

is all of $\mathbb{R}^2$

$\operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\} \subset \mathbb{R}^2$ is also all of $\mathbb{R}^2$

$\cdot$ $\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} \subset \mathbb{R}^3$ is the xy plane $(z=0)$

$\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \subset \mathbb{R}^3$ is $\mathbb{R}^3$

$\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ is also all of $\mathbb{R}^3$

Q: how many vectors do we need, to span $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n, \dots$?

Q: What's going on with the last example?

Linear independence:

$\vec{v_1}, \dots, \vec{v_k}$ are said to be *linearly independent* if the only values of $\lambda_1, \dots, \lambda_k$ that satisfy

$$\lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} + \dots + \lambda_n \vec{v_n} = 0$$

are $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

In other words, $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent if and only if none of the vectors can be written as a linear combination of the others. Indeed, if there was a choice of $\lambda_1, \dots, \lambda_k$ so that they are not all zero and

$$\lambda_1 \vec{v_1} + \dots + \lambda_n \vec{v}_n = 0$$

then we could pull the non-zero one (say λ_5 is $\neq 0$)

$$\lambda_1 \vec{v_1} + \dots + \lambda_4 \vec{v_4} + \lambda_6 \vec{v_6} + \dots + \lambda_n \vec{v_n} = -\lambda_5 \vec{v_5}$$

then $\vec{v_5} = -\dfrac{\lambda_1}{\lambda_5}\vec{v_1} + \dots + \dfrac{-\lambda_4}{\lambda_5}\vec{v_4} + \dfrac{-\lambda_6}{\lambda_5}\vec{v_6} + \dots + \dfrac{-\lambda_n}{\lambda_5}\vec{v_n}$

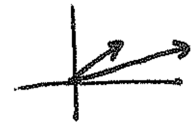
and we would have expressed v_5 in terms of the others (as a linear combination)

<u>examples</u> • $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent.

check: $\lambda_1\begin{pmatrix}1\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = 0 \quad \Rightarrow \quad \lambda_1 = 0 \text{ and } \lambda_2 = 0$$

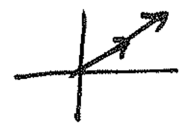
• $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix}$ are independent



<u>check</u>: $\lambda_1\begin{pmatrix}1\\1\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}\lambda_1 + 2\lambda_2\\ \lambda_1 + \lambda_2\end{pmatrix} = 0$

$$\Rightarrow \lambda_1 = 0 \text{ and } \lambda_2 = 0.$$

• $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\2\end{pmatrix}$ are dependent.

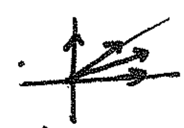


$$(-2)\cdot\begin{pmatrix}1\\1\end{pmatrix} + 1\cdot\begin{pmatrix}2\\2\end{pmatrix} = 0$$

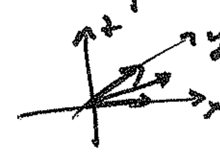
In the examples before:

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are linearly independent

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ are dependent.

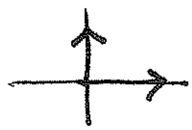


$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ are also dependent.

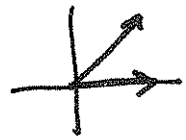


Again: how many do we need to span the whole space? We know now that if $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent we have too many. So we start with a spanning subset, remove vectors so that they still span but are not dependent anymore, what we get is a <u>basis</u>. The point is that once we have a basis $\vec{v}_1, \dots, \vec{v}_n$, then every vector can be expressed <u>uniquely</u> as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

Def: A <u>basis</u> for a vector space is a subset of elements that ~~are~~ span the whole space <u>and</u> are linearly independent.

<u>eg</u>: • $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. 

form a basis

• $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$  also form a basis

. I can write every vector in terms of these

. no redundancy

Indeed, if $\vec{w} = \lambda_1 \vec{v_1} + \dots + \lambda_n \vec{v_n}$

but also $\vec{w} = \mu_1 \vec{v_1} + \dots + \mu_n \vec{v_n}$, then

$$0 = \vec{w} - \vec{w} = (\lambda_1 - \mu_1)\vec{v}_1 + \dots + (\lambda_n - \mu_n)\vec{v_n}$$

so if $\vec{v_1}, \dots, \vec{v_n}$ are linearly independent, then

$$\lambda_1 - \mu_1 = \lambda_2 - \mu_2 = \dots = \lambda_n - \mu_n = 0$$

So if we have a basis $\vec{v}_1, \dots, \vec{v_n}$ then every vector can be written uniquely in terms of these.

How many elements in a basis? for $\mathbb{R}^2$ we saw 2

for $\mathbb{R}^3$ we saw 3

for $\mathbb{R}^n$ n

(look at standard basis $e_1, \dots, e_n$)

The size of a basis for a vector space is the <u>dimension</u> of a vector space (it doesn't depend on the basis you choose, it's always the same number) recall how to multiply a matrix with a vector.

$$2\times 2: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

one thing we do a lot as I explained before: solve $A\vec{x}=\vec{b}$

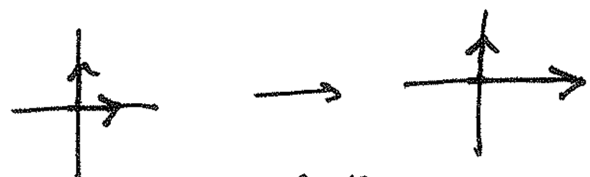
Let's look at some examples:

- $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ this is the id Matrix

  $I\vec{x} = \vec{x}$ for all $\vec{x}$.

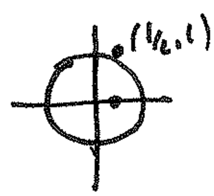
  so $\vec{x} = \vec{b}$

- $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ what does this do?

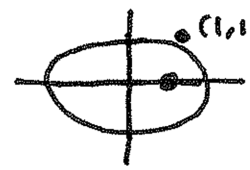


dilates in the x axis direction

so for example $A\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ makes $\vec{x} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$



into



- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

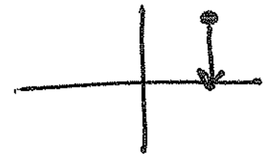
  swaps x and y.

$A\vec{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

unique solution.

- different example

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$



projects onto x axis.

let's look at $A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$x = 1$ $\quad$ y can be anything.

so we have infinitely many solutions.

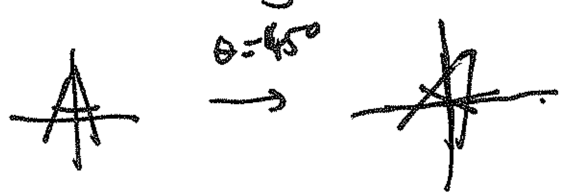
we can see that it's a one parameter family

now look at $A\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ no solutions!

but notice that if there is a solution, it's always a one-parameter family.

we'll come back to this.

- $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ rotation by θ $\quad$ unique solution!



- if I wrote down this with equations

$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ $\qquad$ $\begin{aligned} x + y &= 3 \\ 2x + 3y &= 7 \end{aligned}$

we could play with the equations.

remove 2 times 1st eq from second

$$x + y = 3$$
$$0 + y = 1 \qquad \text{so } y = 1$$

remove second from first:

$$x + 0 = 2 \qquad x = 2.$$
$$0 + y = 1$$

we could also do this with just the numbers to make it easier to write.

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 3 & 7 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right) \qquad x = 2 \quad y = 1.$$

these are row operations: what are we allowed to do?
everything we could do with equations:

- add some multiple of a row to another
- multiply a row by a number (which is not 0)
- swap rows

The point is that we could always go back.

eg: $x_1 - x_2 - x_3 = -3$

$2x_1 + 3x_2 + 5x_3 = 7$

$x_1 - 2x_2 + 3x_3 = -11$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 2 & 3 & 5 & 7 \\ 1 & -2 & 3 & -11 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 2 & 3 & 5 & 7 \\ 0 & -1 & 4 & -8 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 7 & 13 \\ 0 & -1 & 4 & -8 \end{array}\right) \xrightarrow{R_3 \leftarrow 5R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 7 & 13 \\ 0 & -5 & 20 & -40 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 7 & 13 \\ 0 & 0 & 27 & -27 \end{array}\right) \xrightarrow{R_3 \leftarrow \frac{1}{27} R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 7 & 13 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

we see already that $x_3 = -1$

$$\xrightarrow{R_2 \leftarrow R_2 - 7R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 0 & 20 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 \cdot \frac{1}{5}} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

so $x_1 = 0$

$x_2 = 4$

$x_3 = -1$

we also see that this is the only solution.

more about that later.

$$A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix} \qquad A\vec{x} = \vec{0}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 6R_1} \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 0 & 6 & 9 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 6 & 9 \\ 0 & 6 & 9 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 6 & 9 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 2 & -2 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

this is as simple as it will become.

$$2x_1 - x_3 = 0$$
$$2x_2 + 3x_3 = 0$$

you can see that we have a choice here we can pick x_3 for example and the others follow. $x_1 = \frac{x_3}{2}$ $\quad x_2 = -\frac{3}{2}x_3$

so the solution is $\begin{pmatrix} x_3/2 \\ -3/2\, x_3 \\ x_3 \end{pmatrix}$ one parameter family.

Matrices are linear functions and linear functions are matrices.

A linear function $f: \mathbb{R}^n \to \mathbb{R}^m$ is a function that takes n-vectors and gives m-vectors, such that

$$f(v+w) = f(v) + f(w)$$

and

$$f(\lambda v) = \lambda f(v)$$

A matrix gives a linear function by multiplying with vectors. An $m \times n$ matrix gives:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$A\vec{x} = \vec{y}$$

now: $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$

and $A(\lambda\vec{v}) = \lambda A\vec{v}$

so this is a linear function.

On the other hand, every linear function gives a matrix. Say we have a ~~f~~ linear function $f: \mathbb{R}^n \to \mathbb{R}^m$

since f is linear, it suffices to know what f does to the standard basis. Indeed, say $\vec{v} = \lambda_1 e_1 + \cdots + \lambda_n e_n$

then $f(v) = \lambda_1 f(e_1) + \lambda_2 f(e_2) + \cdots + \lambda_n f(e_n)$

then check that if $A = \left(\begin{pmatrix} f(e_1) \end{pmatrix} \begin{pmatrix} f(e_2) \end{pmatrix} \cdots \begin{pmatrix} f(e_n) \end{pmatrix} \right)$

then $f(v) = Av$ (the columns of A are the m-vectors $f(e_i)$)