Penn
University of Pennsylvania

Math 240 Midterm 1

Name: SOLUTIONS

There are 8 questions plus one extra credit question. The questions are weighted differently. Make sure you check all your answers before attempting the extra credit question.

For grading purposes:

1: __________
2: __________
3: __________
4: __________
5: __________
6: __________
7: __________
8: __________
EC: __________

Total: __________

1. (10 points) Solve the system of equations

$$\begin{aligned} x + y + 3z &= 6 \\ x - y + 2z &= 1 \\ 2x + y + 9z &= 13 \end{aligned}$$

$$\boxed{\text{Answer:} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$$

In matrix form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 1 & -1 & 2 & 1 \\ 2 & 1 & 9 & 13 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 1 & -1 & 2 & 1 \\ 0 & -1 & 3 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 0 & -2 & -1 & -5 \\ 0 & -1 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow R_3 \times 2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 0 & -2 & -1 & -5 \\ 0 & -2 & 6 & 2 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 7 & 7 \end{array}\right) \xrightarrow{R_3 \leftarrow \frac{R_3}{7}} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \leftarrow R_1 - 3R_3]{R_2 \leftarrow R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

2. (10 points) Find the function $y(x)$ that satisfies

$$x^2y'' - 6xy' + 12y = 0$$

with $y(1) = 0$ and $y'(1) = 2$

$$\boxed{y(x) = -2x^3 + 2x^4}$$

Cauchy-Euler equation. Try $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$x^2 m(m-1)x^{m-2} - 6x\, m x^{m-1} + 12x^m = 0$

$x^m(m2 - m - 6m + 12) = 0$

$x^m(m^2 - 7m + 12) = x^m(m-3)(m-4)$

so $m = 3$ or 4

$y(x) = c_1 x^3 + c_2 x^4$

$y'(x) = 3c_1 x^3 + 4c_2 x^4$

$y(1) = 0 \Rightarrow c_1 + c_2 = 0$

$y'(1) = 2 \Rightarrow 3c_1 + 4c_2 = 2$

$\Rightarrow c_2 = 2$, $c_1 = -2$

$y(x) = -2x^3 + 2x^4$

3. (15 points) Find the general solution to the differential equation

$$\frac{d^4y}{dx^4} - 7\frac{d^3y}{dx^3} + 12\frac{d^2y}{dx^2} = 6$$

$$\boxed{y(x) = C_1 + C_2 x + C_3 e^{3x} + C_4 e^{4x} + \frac{1}{4}x^2}$$

Linear equation with constant coefficients.

Try $y = e^{mx}$ $\quad y' = me^{mx}, \quad y'' = m^2 e^{mx}, \quad y''' = m^3 e^{mx}, \quad y^{(4)} = m^4 e^{mx}$

$m^4 e^{mx} - 7m^3 e^{mx} + 12m^2 e^{mx} = 0$ (homogeneous version of the equation)

$e^{mx}(m^4 - 7m^3 + 12m^2) = 0$

$e^{mx} m^2 (m^2 - 7m + 12) = 0$

$e^{mx} m^2 (m-3)(m-4) = 0$

$m_1 = m_2 = 0$ is a double root. $m_3 = 3 \quad m_4 = 4$

So $y_c(x) = C_1 e^{0x} + C_2 x e^{0x} + C_3 e^{3x} + C_4 e^{4x}$

$= C_1 + C_2 x + C_3 e^{3x} + C_4 e^{4x}$

For y_p, try $y = Ax^2 \quad y' = 2Ax \quad y'' = 2A \quad y''' = 0 \quad y'''' = 0$

Plug in: $0 - 7 \cdot 0 + 12 \cdot 2A = 6$

$A = \frac{1}{4} \qquad y_p = \frac{1}{4}x^2$

$y = y_p + y_c = \frac{1}{4}x^2 + C_1 + C_2 x + C_3 e^{3x} + C_4 e^{4x}$

4 (10 points) Find $\det(A^{-1}BA)$ for

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -5 & -2 & 0 \\ 3 & -5 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 2 & 3 \\ -4 & -2 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

$\det(A^{-1}BA) =$

We have $\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A)$

$$= \det(A^{-1})\det(A)\det(B)$$

$$= \det(\underbrace{A^{-1}A}_{I}) \cdot \det(B) = \det(B)$$

$$\det(B) = 1.((-2)2 - 0.(-1)) - 2((-4)2 - 0.3) + 3((-4)(-1) - (-2)3)$$

$$= -4 + 16 + 30 = 42$$

5. (12 points) Find the solution to the system of differential equations

$$\frac{dx}{dt} = 2x - y$$
$$\frac{dy}{dt} = 2x + 5y$$

that satisfies $x(0) = 0$, $y(0) = 1$.

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{4t}}$$

in matrix form: $X' = AX \quad X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad A = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$

Find eigenvalues and eigenvectors

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 \\ 2 & 5-\lambda \end{pmatrix} = (\lambda - 3)(\lambda - 4)$$

<u>for $\lambda = 3$</u>: $A - 3I = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad v_1 = -v_2 \quad K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

<u>for $\lambda = 4$</u>: $A - 4I = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad 2v_1 = -v_2 \quad K_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

So $X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{4t}$

$X(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \quad c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So $\begin{matrix} c_1 - c_2 = 0 \\ -c_1 + 2c_2 = 1 \end{matrix} \Rightarrow \quad c_1 = 1$ and $c_2 = 1$

6. (15 points) Find the *general solution* to the system of differential equations given for functions $x(t)$ and $y(t)$ by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \right)}$$

$$\det(A-\lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (\lambda-2)^2$$

$\lambda = 2$ is an eigenvalue with multiplicity two.

$$A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ is the only eigenvector.

so $$X = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{2t} + P e^{2t} \right)$$

where $(A-2I)P = K_1$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{so} \quad P = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

7. (18 points) TRUE or FALSE? You do not need to justify your answers

(i) The $n \times n$ matrix whose entries are all equal to 1 has rank n.

(ii) For an $n \times n$ matrix A, if $\det(A) = 0$, the the equation $A\vec{v} = 0$ has infinitely many solutions.

(iii) If D is an $n \times n$ diagonal matrix and A is any $n \times n$ matrix, then $AD = DA$, that is, A and D commute.

(iv) For every two $n \times n$ diagonal matrices D_1 and D_2, we have $D_1 D_2 = D_2 D_1$. i.e. they commute.

(v) If two 3×3 matrices A and B are diagonalizable with the same eigenvectors, then they commute.

(vi) Let A be a 3×3 matrix with eigenvalues 1, 2 and 0. Then A is diagonalizable.

(vii) Let A be a 3×3 matrix with eigenvalues 1, 2 and 0. Then A is invertible.

(viii) The subset $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x_1 - 3x_2 + x_3 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

(ix) The subset $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 x_2 = 0 \right\}$ is a subspace of $\mathbb{R}^2$.

(i) Such a matrix has all its rows $(1\ 1\ 1 \cdots 1)$ so the rank is 1 not n. FALSE

(ii) $\det A = 0 \iff A$ is singular $\iff \mathrm{rk} A < n \iff \mathrm{nullity} A > 0$ TRUE

(i)	○ true	● false
(ii)	● true	○ false
(iii)	○ true	● false
(iv)	● true	○ false
(v)	● true	○ false
(vi)	● true	○ false
(vii)	○ true	● false
(viii)	● true	○ false
(ix)	○ true	● false

(iii) FALSE $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

(iv) true $\begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots\ d_n \end{pmatrix}\begin{pmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots\ b_n \end{pmatrix} = \begin{pmatrix} d_1 b_1 & & 0 \\ & d_2 b_2 & \\ 0 & & \ddots\ d_n b_n \end{pmatrix}$

(v) true. $A = PD_1P^{-1}$, $B = PD_2P^{-1}$ $AB = PD_1P^{-1}PD_2P^{-1} = PD_1D_2P^{-1} = PD_2D_1P^{-1} = PD_2P^{-1}PD_1P^{-1}$ $= BA$

(vi) three distinct eigenvalues gives three independent eigenvectors. TRUE

(vii) 0 is an eigenvalue so there is a $\vec{v}$ s.t. $A\vec{v} = 0\vec{v} = 0$ so $\mathrm{nullity} A > 0$. FALSE

(viii) TRUE this is a plane through the origin in $\mathbb{R}^3$.

(ix)  this is the union of the x axis and the y axis. not closed under addition $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in the subset.

8. (10 points) Find the *general solution* to the differential equation

$$x^2y'' + xy' = 0$$

$y(x) =$ $c_1 + c_2 \ln x$

Cauchy-Euler equation. $y = x^m$

$m(m-1) + m = 0$ is the auxiliary equation.

$m^2 - m + m = m^2 = 0$

$m = 0$ is a double root.

$y = c_1 x^0 + c_2 x^0 \ln x$

$= c_1 + c_2 \ln x$

Extra Credit (15pts) Your answers must include complete justification. Please write sentences to explain your proof. Each part is 5 points, no partial credit will be given:

Recall that the image of an $n \times n$ matrix A is:

$$\operatorname{Im} A = \left\{\vec{w} \in \mathbb{R}^n \,\middle|\, \vec{w} = A\vec{v} \text{ for some } \vec{v} \in \mathbb{R}^n\right\}$$

The *kernel* of A is the solution space to $A\vec{v} = 0$.

$$\operatorname{Ker} A = \left\{\vec{v} \in \mathbb{R}^n \,\middle|\, A\vec{v} = 0\right\}$$

a) Show that, if there is an $n \times n$ matrix A whose image is identical to its kernel, i.e. $\operatorname{Im} A = \operatorname{Ker} A$, then n must be even.

b) Give an example of such a matrix.

c) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Think about A mapping the plane to the plane. In other words, think about the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ given by multiplying by A. What is the area of the image of the unit square $\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 0 < x < 1 \text{ and } 0 < y < 1\right\}$ under this linear transformation. You might need to use the fact that

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin \alpha$$

where α is the angle between $\vec{v}$ and $\vec{w}$. And if you forgot how to calculate cross products, you might need to use that

$$|\vec{v} \times \vec{w}| = v_1 w_2 - v_2 w_1$$

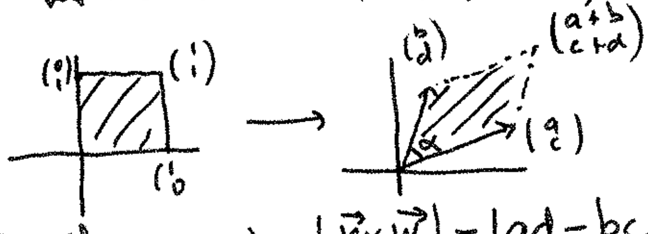
for $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$.

a) rank A = dimension of the image of A
nullity = dimension of the kernel of A so if ~~rank~~ $\operatorname{Im} A = \operatorname{Ker} A$
then rank A = nullity of A so $\underbrace{\text{rank } A + \text{nullity } A}_{= n}$ is even
so n is even

b) project onto x axis, then rotate by $90°$ counterclockwise
this takes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\operatorname{Ker} A = \operatorname{Im} A$ is the y axis

c) This matrix takes the unit square to the parallelogram whose vertices are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} a+b \\ c+d \end{pmatrix}$ Say $\vec{v} = \begin{pmatrix} a \\ c \end{pmatrix}$, $\vec{w} = \begin{pmatrix} b \\ d \end{pmatrix}$



The area of the parallelogram is base × height. Base is $|\vec{v}|$, height is $|\vec{w}| \cdot \sin\alpha$.

So the area is $|\vec{v} \times \vec{w}| = |ad - bc|$