RESEARCH STATEMENT

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I wrote my first papers on algebraic geometry; specifically on derived categories of sheaves, matrix factorizations and projective duality. I am now studying the relationship between computational complexity theory on the one hand, and algebraic geometry, category theory, and homological algebra on the other.

My research is based on three basic theses:

(1) Every kind of algebraic or geometric object in mathematics has a natural notion of computational complexity.
(2) The complexity of a mathematical object has something to do with its relationships with other objects, and with its invariants.
(3) Concepts like 'polynomial vs. exponential complexity', 'complexity classes', 'completeness', 'reduction' are more pervasive in mathematics than it would seem at first glance.

Based on these, I am studying computational complexity classes in algebra, geometry and category theory that exist *internally* in these fields. I see my research as an effort to make computational complexity a proper part of mainstream mathematics.

In theoretical computer science, it is widely expected that NP-complete problems like the Boolean satisfiability problem have exponential complexity lower-bounds, yet the best known such bound is $n^{\sqrt{3}}$ (and this includes some extra space restrictions), so we need as many intermediate problems as we can find to help make progress on these fundamental lower-bound problems. The questions about these geometric and categorical complexity classes are seemingly easier versions of the P vs NP problem. In this sense, my research creates problems of interest to theoretical computer science, as well as a richer mathematical theory around computational complexity theory.

1. Previous Work

1.1. Complexity Classes and Completeness in Algebraic Geometry.

What is the complexity of an algebraic variety? Do there exist natural complexity classes of sequences of algebraic varieties? How do we express complexity results about algebraic varieties without assuming integer, or rational coefficients? These questions were the starting points of my work on complexity classes in algebraic geometry [Isi16].

In recent years, there has been renewed interest from the mathematical community in Valiant's algebraic/arithmetic complexity classes of polynomials, and the determinant vs permanent problem which is an algebraic problem similar to the P vs NP problem. In *loc. cit.*, I described an algebraic geometry version of Valiant's theory, showed that there is a version of the P vs NP question in it, and proved that there is an NP-complete problem in this theory.

The complexity of an embedded projective or bi-projective algebraic variety X is defined as the total cost of producing polynomials that cut out X, using a straight-line program (a list of arithmetic instructions). A sequence of varieties (X_n) is in $\mathrm{P}_{\mathrm{proj}}$ if the complexity of X_n grows polynomially. Sequences in $\mathrm{NP}_{\mathrm{proj}}$ are the projections of sequences of $X_n \subset \mathbb{P}^{\mathrm{n}_1(n)} \times \mathbb{P}^{\mathrm{n}_2(n)}$ in $\mathrm{P}_{\mathrm{proj}}$, onto the first component $\mathbb{P}^{\mathrm{n}_1(n)}$. These are natural geometric generalizations of P and NP from classical Turing Complexity.

Key concepts in computational complexity are those of reduction and completeness. A sequence $(X_n \subset \mathbb{P}^{\mathrm{n}(n)})$ reduces to $(Y_n \subset \mathbb{P}^{\mathrm{m}(n)})$ if we can get each X_n from a Y_m by taking a linear section (m possibly larger than n but polynomially bounded in n). A sequence Y_n is complete in a complexity class of every other sequence in in the class reduces to (Y_n); making Y_n the sequence with the highest complexity in its class. The main result of the paper is the following:

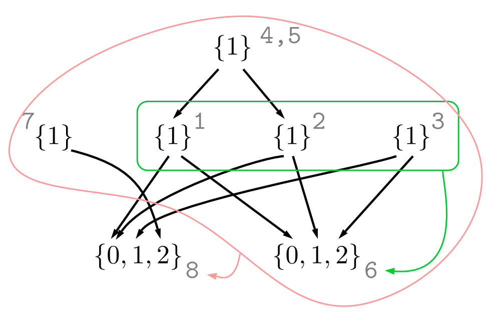
Theorem 1. *The Segre embedding of the Universal Circuit Resultant is* $\mathrm{NP}_{\mathrm{proj}}$*-complete.*

It can be said that this Universal Circuit Resultant is 'the Travelling Salesman Problem, or the Boolean Satisfiability Problem of this theory'. While this is true in the sense that it is a complete problem, the more natural question is the completeness of the resultant (of, say, a number of quadratic equations). Indeed, the NP-completeness of the usual resultant (the so-called 'homogeneous Nullstellensatz problem') is a long-standing open problem in Blum-Shub-Smale theory [Shu14]. In fact, it was not known whether any sequence of compact objects could be NP-complete. The proof in [Isi16] (easily adapted to the Blum-Shub-Smale setting) makes the Universal Circuit Resultant the first compact sequence known to be NP-complete.

This notion of complexity leads to many basic questions in algebraic geometry. For example: given a variety X_n, what is the highest possible complexity of a desingularization of X_n as a function of the complexity of X_n? In the long run, future analyses on the complexity effects of interesting geometric constructions such as desingularization, taking minimal models, taking projective duals, and future results on the complexity implications of invariants should use this language rather than using the Turing Machine language (and being force to use integer or rational coefficients), or using the dense or sparse models within BSS complexity.

1.2. **Categorical Complexity.** We can find a good complexity theory within each category we consider, but how do we relate the complexities of objects that are in different categories but are somehow related? For example, we want to try to analyze the complexity effects of the cohomology functor, or certain derived functors. It is apparent that we need a general theory of complexity that applies equally well to algebras, algebraic varieties, vector spaces, topological spaces,... and all kinds of mathematical objects.

With this thought, S. Basu and I started analyzing complexity theoretic properties of toposes in 2015, and ended up discovering a simple approach that makes sense for all categories, not just toposes. This is written in [BI16]. The idea is to pre-determine a set of morphisms in the category called the *basic morphisms*; and use constructions called *diagram computations* to make new diagrams in the category. One starts each diagram computation with an empty diagram, and then successively adds new basic morphisms, or the limits and colimits of subdiagrams to make more and more complex objects and morphisms. For example, here is a diagram computation of the morphism of sets $f : \{0,1,2\} \to \{0,1,2\}$, $f(0) = 0$, $f(1) = 0$, $f(2) = 1$ using two colimits.



1. _,$\{1\} \xrightarrow{\mathrm{id}} \{1\}$,1
2. _,$\{1\} \xrightarrow{\mathrm{id}} \{1\}$,2
3. _,$\{1\} \xrightarrow{\mathrm{id}} \{1\}$,3
4. _,$\{1\} \xrightarrow{\mathrm{id}} \{1\}$,1
5. 4,$\{1\} \xrightarrow{\mathrm{id}} \{1\}$,2
6. colim(1,2,3)
7. _,$\{1\} \xrightarrow{\mathrm{id}} \{1\}$,7
8. colim(1,2,3,4,6,7)

The total number of lines determines the cost of the computation and leads to the definition of complexity, which is the cost of the cheapest diagram computation that produces an object or morphism. Diagram computations come in three kinds, one where you are only allowed to use so-called *constructive limits*, one with only *constructive colimits*, and the full *mixed computations*; leading to the notions of limit complexity, colimit complexity and mixed complexity.

Here are some of the highlights from [BI16]:

Basic examples: For sets, the colimit complexity of a set S is $|S|+1$. Infinite sets are 'non-computable' in this theory. For topological spaces, colimit computations starting from simplices and face maps define a simplicial complexity for topological spaces. Similarly, mixed computations starting from points and intervals make cubical complexity. These measure how hard it is to make a space from simplices and cubes respectively. Another important basic example is the lattice of subsets of a finite set (considered as a category with inclusions as the morphisms). This corresponds precisely to monotone Boolean complexity.

Comparison theorems for varieties/schemes: Starting from basic morphisms $\mathbb{A}^1 \xrightarrow{c} \mathbb{A}^1$, $\mathbb{A}^2 \xrightarrow{+,\times} \mathbb{A}^1$, $\mathbb{A}^2 \xrightarrow{\pi_1,\pi_2} \mathbb{A}^1$, and $\mathbb{A}^1 \to *$, and $* \xrightarrow{c} \mathbb{A}^1$ in the category of algebraic varieties over a field k, we get a notion of geometric complexity $C(X)$ for any variety X. Indeed, the complexity function C has the same value for isomorphic varieties, and will be low for a hypersurface whose geometry is simple, even if the defining polynomial has high complexity.

Nonetheless, there is a comparison theorem in each direction: If X is a variety which is the 0-set of a polynomial with arithmetic complexity N, then the categorical limit complexity of X is bounded by $5N$. On the other hand, if X is a variety with limit complexity N, then X is *isomorphic to* the zero-set of a polynomial map with complexity linearly bounded in N, i.e. it is less than $c_0 + c_1 N$ where c_0 and c_1 are globally fixed constants.

For projective schemes in $\mathbb{P}^n$: by building affine pieces with limits and then glueing them using colimits, we showed that the mixed complexity of a projective scheme is bounded above by a constant multiple of $n^2 N$, where N is the arithmetic complexity of its defining equations.

Comparison theorem for modules over polynomial rings: The category of $k[x_1,\ldots,x_n]$-modules is important because it is a step on the way to a full theory for sheaves. Moreover, we can think of quotient modules as sheaves supported on embedded subvarieties, and therefore have a theory that is closer to Valiant complexity. Here, colimit complexity is considered with basic morphisms $R \xrightarrow{x_i} R$ for variables, $R \xrightarrow{c} R$ for constants $c \in k$, the inclusions $R \xrightarrow{i_1,i_2} R \oplus R$, the diagonal map $R \xrightarrow{\Delta} R \oplus R$, addition $R \oplus R \xrightarrow{+} R$, and the canonical map $R \to \{0\}$.

We proved the following comparison theorems. If $f \in k[x_1,\ldots,x_n]$ has arithmetic formula complexity N, then $R \xrightarrow{f} R$ has categorical complexity linearly bounded in N. On the other hand, if $R \xrightarrow{f} R$ has colimit complexity N, then the arithmetic circuit complexity of f is polynomially bounded in N.

P vs NP problem in categories: For any category $\mathcal{C}$ and set of basic morphisms, consider the morphism category $\mathcal{C}^{\bullet\to\bullet}$. Categorical complexity gives a function on the objects of $\mathcal{C}^{\bullet\to\bullet}$. Now consider the full subcategory of monomorphisms in $\mathcal{C}^{\bullet\to\bullet}$, a left-adjoint to the inclusion of this subcategory is called the image functor. We can consider it as a functor

$$\mathrm{im}_{\mathcal{C}} : \mathcal{C}^{\bullet\to\bullet} \to \mathcal{C}^{\bullet\to\bullet}.$$

Then, we can pose the following question. If a morphism $X \xrightarrow{f} Y$ has categorical complexity N, is the complexity of $\mathrm{im}_{\mathcal{C}}(X \xrightarrow{f} Y)$ polynomially bounded in N? This is the analogue of the P vs NP problem in the category $\mathcal{C}$. This can be studied for the any category $\mathcal{C}$ where the image functor exists; in particular, categories of projective varieties/schemes, and semi-algebraic or constructible spaces. We studied it in detail in the category of $k[x_1,\ldots,x_n]$-modules, and found an upper bound for the complexity of the image functor there.

1.3. The σ-Model-Landau-Ginzburg-Model correspondence.

In earlier work, [Isi13], I proved a statement relating the derived category of any variety given as the zero-scheme of a regular section of a vector bundle to the singularity category of a related singular space. This statement, based on Koszul duality, provides a new derived equivalence bridge between varieties and Landau-Ginzburg models.

Theorem 2. [Isi13] *Let X be a smooth variety and Y be the zero-scheme of a regular section $s \in \Gamma(X, \mathcal{E})$ of a finite rank vector bundle $\mathcal{E}$. There is an equivalence*

$$\mathrm{D^b}(Y) = \mathrm{D}^{\mathbb{C}^\times}(\mathrm{Tot}\,\mathcal{E}^*, < s, \cdot >)$$

More precisely, I proved that the $\mathbb{C}^\times$-equivariant singularity category of the only singular fiber of the function given by pairing with s is equivalent to the derived category of Y as a triangulated category. The proof uses a very elegant Koszul duality statement due to Mirkovic and Riche, called linear Koszul duality. This equivalence also holds for the natural dg-enhancements of these categories by the general results of Lunts and Orlov [LO10] about uniqueness of enhancements.

With this result, you can start with a variety Y which a complicated derived category, but after replacing it with the derive-equivalent pair $(\operatorname{Tot}\mathcal{E}^*/\mathbb{C}^*, < s, \cdot >)$, you have much more freedom in geometrically manipulating $\operatorname{Tot}\mathcal{E}^*$ since it is a much simpler space. This is the basis of a lot of application of the results on variations of GIT quotients and derived categories [BFK12, H-L12].

1.4. **Homological Projective Duality, Relative Orlov Theorem, Matrix Factorizations.** My collaboration with M. Ballard, D. Deliu, D. Favero and L. Katzarkov. In [BDFIK13] and [BDFIK14], we studied Homological Projective Duality for quotient varieties. Introduced by Kuznetsov, this homological phenomenon is related to projective duality and can be used to prove a number of derived equivalences originally conjectured by physicists. We invented a general new method for constructing homological projective duals of quotient varieties and we used it to study this phenomenon for d-Veronese embeddings in detail. In another paper, [BDFIK12], we developed essential technical tools for working with matrix factorization categories and applied these new results to lift fully faithfulness statements about derived categories to matrix factorization categories.

In the study of the derived categories of algebraic varieties, a useful notion is that of a semi-orthogonal decomposition. Such a decomposition is a way of dissecting a triangulated category into smaller pieces, with the sets of morphisms between the pieces forming the analogue of glueing data. Sometimes, these decompositions have a clear geometric in origin (e.g. for blow-ups, projective bundles, and variations of GIT quotients). Other times, the decompositions are of a more mysterious nature. For example, the derived category of a certain Pfaffian cubic 4-Fold has a semi-orthogonal decomposition in which one of the pieces is equivalent to the derived category of a K3-surface.

In [Kuz07], Kuznetsov discovered a very interesting duality phenomenon about semi-orthogonal decompositions called Homological Projective Duality. It involves semi-orthogonal decompositions of the derived category $\mathrm{D^b}(X)$ of a smooth scheme X together with a morphism $X \to \mathbb{P}(V)$ and the derived category of a homological projective dual (HPD) $Y \to \mathbb{P}(V^*)$. The basic assumption is that $\mathrm{D^b}(X)$ has a special kind of semi-orthogonal decomposition, called a *Lefschetz decomposition*, of the form:

$$\mathrm{D^b}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_i(i) \rangle,$$

where $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots \supset \mathcal{A}_i$ is a filtered sequence of subcategories. The main virtue of a Lefschetz decomposition is that it behaves well with respect to taking hyperplane sections of X. This gives a semi-orthogonal decomposition of the family $\mathcal{X}$ of all hyperplane sections of X. Y is said to be HPD to X if this decomposition is of the form:

$$\mathrm{D^b}(\mathcal{X}) = \langle \mathrm{D^b}(Y), \mathcal{A}_1(1) \boxtimes \mathrm{D^b}(\mathbb{P}(V^*)), \ldots, \mathcal{A}_i(i) \boxtimes \mathrm{D^b}(\mathbb{P}(V^*)) \rangle, \tag{1}$$

and the embedding of $\mathrm{D^b}(Y)$ is a Fourier-Mukai functor relative to $\mathrm{P}(V^*)$.

Once this relationship is established between $X \to \mathrm{P}(V)$ and a possibly non-commutative $Y \to \mathrm{P}(V^*)$ then not only do we have that Y is smooth and has a *dual Lefschetz decomposition* which makes X HPD to Y but also we have semi-orthogonal decomposition relationships between the derived categories of any complete linear sections of X and the corresponding dual complete linear sections of Y. We call this Kuznetsov's *Fundamental Theorem of HPD.* This enables one to prove the existence of many interesting semi-orthogonal decompositions in algebraic geometry, including the one for the Pfaffian cubic mentioned above.

Theorem 3 (Kuznetsov - *Fundamental Theorem of HPD*)**.** *Let $Y \to \mathrm{P}(V^*)$ be a homological projective dual to $X \to \mathrm{P}(V)$ with respect to the Lefschetz decomposition $\{\mathcal{A}_i\}$. Let $N = \dim V$. The following are true:*

- *The category* $\mathrm{D^b}(Y)$ *admits a dual Lefschetz decomposition*

$$\mathrm{D^b}(Y) = \langle \mathcal{B}_j(-j), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle$$

.

- *Assume that* X_L *and* Y_L *are complete linear sections, i.e.*

$$\dim(X_L) = \dim(X) - r \textit{ and } \dim(Y_L) = \dim(Y) + r - N.$$

  Then there exist semi-orthogonal decompositions,

$$\mathrm{D^b}(X_L) = \langle \mathcal{C}_L, \mathcal{A}_r(1), \dots, \mathcal{A}_i(i - r + 1) \rangle,$$

  and,

$$\mathrm{D^b}(Y_L) = \langle \mathcal{B}_j(N - r - j - 1), \dots, \mathcal{B}_{N-r}(-1), \mathcal{C}_L \rangle.$$

Our results in [BDFIK13] start with the basic observation that if X is a (stacky) GIT quotient $Q^{\mathrm{ss}}(\mathcal{L})/G$ that has a simple variation of GIT quotients, then a Lefschetz decomposition for X with respect to the morphism corresponding to $\mathcal{L}$ can be constructed using the variation of GIT quotients results in [BFK12, H-L12].

The second and crucial observation is that when $\mathcal{X}$ is replaced by a derived-equivalent Landau-Ginzburg pair $(\mathcal{Y}, w)$ using the σ-LG correspondence (Theorem 2 above) from [Isi13], there is an induced variation of GIT quotients on $\mathcal{Y}$ that gives the decomposition (1). One therefore obtains a pair $(\mathcal{Y}', w)$ which can be considered as a homological projective dual. Our main result is as follows:

Theorem 4. *The conclusions of the Fundamental Theorem of HPD hold for* $(\mathcal{Y}', w)$.

Using the σ-LG correspondence or a more general form of Koszul duality as necessary on $(\mathcal{Y}', w)$ allows one to replace the Landau-Ginzburg pair with a non-commutative variety. This is done in [BDFIK14]. The conclusion is that many HPD statements come from considering variations of GIT quotients *at the LG-level.* This story fits nicely into the following diagram.

$$\begin{tikzcd}
\mathrm{D}(\mathcal{Y}, w) \arrow[r, leftrightarrow, "\text{VGIT}"] \arrow[d, leftrightarrow, "\sigma\text{-LG corr.}"'] & \mathrm{D}(\mathcal{Y}', w) \arrow[d, leftrightarrow, "\text{Koszul-Duality}"] \arrow[dl, leftrightarrow, dashed, "\text{Relative Orlov}"'] \\
\mathrm{D^b}(\mathcal{X}) & \mathrm{D^b}(Y) \arrow[l, "\text{HPD}"']
\end{tikzcd}$$

In the case when $X = \mathrm{P}(W)$ considered as embedded in $\mathrm{P}(S^d W)$ via the d-Veronese embedding, the decomposition (1) is a new, relative version of a well-known theorem of Orlov [Orl09].

The rest of [BDFIK14] focuses on implementing the right side of the diagram in this d-Veronese case. We consider a 'local generator' of the matrix factorization category of the pair $(W \times \mathbb{P}(S^d W^*), w)$. Calculating its graded sheaf-endomorphism algebra over $\mathbb{P}(S^d W^*)$ and using homological perturbation techniques, we obtain a sheaf $\mathcal{A}$ of $\mathcal{A}_\infty$-algebras over $\mathbb{P}(S^d W^*)$. When $d > 2$, we have

$$\mathcal{A} = \mathrm{Sym}(u\mathcal{O}_{\mathbb{P}(S^d W^*)}(1), u^{-1}\mathcal{O}_{\mathbb{P}(S^d W^*)}(-1)) \otimes \Lambda^\bullet W^*,$$

where

$$\mu^d(1 \otimes v_{i_1}, \dots, 1 \otimes v_{i_d}) = \frac{u}{d!} \frac{\partial^d w}{\partial x_{i_1} \dots \partial x_{i_d}}$$

and $\mu^i = 0$ for $2 < i < d$. This gives a different description of the HPD. When $d = 2$, we recover the HPD obtained in [Kuz05].

2. Future Work

Further development of categorical complexity: A short term aim for categorical complexity is to repeat what we did for algebraic varieties for semi-algebraic and constructible spaces: give good comparison theorems with other notions of complexity. The second is to make a satisfying theory for constructible and coherent sheaves, and to see what we can do for complexes of sheaves/modules using the same diagram computation concept with homotopy limits and colimits. The sheaves and derived category questions were actually part of the original motivation for [BI16] but we ended up deciding that more basic development was necessary. Another part I want to think about is the notion of universal computation for diagram computations.

Besides these plans, I am hoping that I can rely on the response from the community to see which parts of categorical complexity need to be developed more than others.

Complexity classes from invariants in algebraic geometry, a continuation of [Isi16]: Let $M(\cdot)$ be an invariant that associates a yes-no answer to each algebraic variety (e.g. is it rational? is it non-empty? is $h^1(X) = 0$). We can associate to M a complexity class as follows. When defining $\mathrm{NP}_{\mathrm{proj}}$, we had projected down efficiently-computable biprojective varieties from $\mathbb{P}^n \times \mathbb{P}^m$; now, instead of taking the points whose fibers are non-empty, we can take the closure of the set of points whose fibers have $M(X_x)$ hold true. So, if M measures whether X is non-empty, then we get $\mathrm{NP}_{\mathrm{proj}}$. In general, we get a new complexity class CC_M associated to each invariant M.

This is a good way to think about the complexity of invariants. Similar to the main theorem of [Isi16], I am able to prove that there is a CC_M-complete sequence.

Complexity of coherent sheaves I can write down a complexity theory of coherent sheaves on $\mathbb{P}^n$ similar to the complexity of projective varieties. Initially, this would be without complexes or derived functors, and in this case, using a method similar to [Isi16], I can show that there is a complete family.

Derived functors and the six functor formalism actually appear very naturally in the context of a better complexity theory of sheaves: the base change theorem ensures that NP (which is defined by push-forward) is closed under reductions (but I don't yet have a completeness theorem in this derived setting).

A construction similar to the one described above associating complexity classes to invariants gives a complexity class CC_H for any vector-space-valued invariant H, e.g. cohomology. I am planning to explore this notion and how it relates to the properties of the invariant H.

Complexity-invariants, projective duality and the determinant vs permanent problem: The most useful 'complexity-invariant' would be an invariant that would: (i) be low for the nth determinant, (ii) be high for the nth permanent, and (iii) would behave well with respect to linear sections. Of course, many cohomological invariants have Lefschetz theorems but few have been studied in relation to the determinant vs permanent problem. This is partly because of how singular these varieties are.

I think invariants of the *projective dual variety* are good to look into for finding invariants like this. Already, looking at the dual side-steps the singularity issue of the determinant, since the dual of the determinant is smooth (it's the image of the Segre embedding). Two other reasons for looking at duals: there is evidence that the invariants of the projective dual may behave well with respect to a kind of 'padding' operation which makes different determinants and permanents have the same degree; also, the dual variety of a reduction by projection (the key step in arguments about linear reductions) is the dual variety of a corresponding linear section.

Some time ago, I began an investigation of the *co-degree* of a variety, i.e. the degree of the projective dual. I calculated some early examples and also did some numerical-algebraic-geometry calculations but the outcome was inconclusive. There is a lot on projective duality I have not considered, as well as interesting developments like the work of Aluffi on the Chern-Mather involution and generalizing known formulas for duality to singular cases [Alu16].

Categorical complexity, lambda-calculus and 2-categories: The Curry-Howard-Lambek correspondence is a beautiful correspondence between Cartesian-closed categories, constructive proof systems and typed lambda calculi. At the level of complexity, it should relate complexity in lambda calculus (which corresponds to Turing complexity, but is not as clean), to proof complexity, to a notion of complexity in Cartesian-closed categories. I would like to compare this with our categorical complexity. There is also a 2-categorical version of the correspondence, which is not fully written-down, but from what I understand, complexity concepts could be more transparent in this 2-categorical version.

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